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On the failure of Lebesgue's criterion for the Summability (C1) of the Fourier series of a function at a point where the function has a discontinuity of the second kind.

BY

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The object of the present paper is to formulate a definite answer to the question : Does Professor Lebesgue's criterion for the summability (C1) of the Fourier series of a function $f(x)$ enable us to settle the question of summability at a point x_0, where $f(x)$ has a discontinuity of the second kind, so that

$$f(x_0+2t)+f(x_0-2t)=\cos\psi(t) \text{ or } \chi(t)\cos\psi(t),$$

$\chi(t)$ and $\psi(t)$ being both monotone functions which tend to infinity as t tends to 0 ? The answer to the question is, that Lebesgue's criterion is *not* satisfied in any case in which the series is summable (C1).

The investigation of the differentiability of

$$F(z)\equiv\int_0^z |f(x_0+2t)+f(x_0-2t)| \, dt$$

at $z=0$ is taken up first, and it is proved that, in every case in which

$$F_1(z)\equiv\int_0^z \{f(x_0+2t)+f(x_0-2t)\}dt$$

is differentiable, $F'(0)$ exists and is *different* from zero. It is next pointed out what exactly Lebesgue's criterion is. Then I state that, as

shown in my previous papers, whenever the Fourier series is summable (C1) at x_0, *the sum is zero.* The failure of Lebesgue's criterion follows at once from the two results stated above.

It is believed that the question considered in this paper has not been considered by any previous writer.

§1

Case: $f(x_0+2t)+f(x_0-2t)=\cos\psi(t)$

When $F'_1(0)$ exists, $F'(0)$ exists and is different from zero.

Proof :—

It has been proved by me in a previous communication* that $F'_1(0)$ is non-existent when

$$\psi(t) \preceq \log\frac{1}{t^2},$$

and that $F'_1(0)$ exists when

$$\psi(t) \succ \log\frac{1}{t^2}.$$

It is obvious that $F'(0)$ is non-existent when $F'_1(0)$ is non-existent. Therefore we need consider, for the purpose of the proof, only the case in which

$$\psi(t) \succ \log\frac{1}{t^2}.$$

Now let t_r denote a quantity such that

$$\psi(t_r)=r\pi+\tfrac{\pi}{2},\ r \text{ being a positive integer;}$$

also let

$$t_r \leq z < t_{r-1}.$$

Then

$$F(z)\equiv\int_0^z |\cos\psi(t)|\,dt=(-1)^r\left\{\int_{t_r}^{z}\cos\psi(t)\,dt-\right.$$

$$\left.\int_{t_{r+1}}^{t_r}\cos\psi(t)dt+\int_{t_{r+2}}^{t_{r+1}}\cos\psi(t)\,dt-\ldots\text{infinity}\right\}$$

* "On the fundamental theorem of the Integral Calculus" (This *Bulletin*, Vol. 16)

$$=(-1)^r\Big[\Big\{\Big(\frac{1}{\psi'}\sin\psi\Big)_{t_r}^{z}-\int_{t_r}^{z}\Big(\frac{d}{dt}\cdot\frac{1}{\psi'}\Big)\sin\psi dt\Big\}$$

$$-\Big\{\Big(\frac{1}{\psi'}\sin\ \psi\Big)_{t_{r+1}}^{t_r}-\int_{t_{r+1}}^{t_r}\Big(\frac{d}{dt}\frac{1}{\psi'}\Big)\sin\ \psi dt\Big\}$$

$$+\Big\{\Big(\frac{1}{\psi'}\sin\ \psi\Big)_{t_{r+2}}^{t_{r+1}}-\int_{t_{r+2}}^{t_{r+1}}\Big(\frac{d}{dt}\frac{1}{\psi'}\Big)\ \sin\psi\ dt\Big\}-\ldots\text{to infinity}\Big]$$

$$=(-)^r\frac{\sin\ \psi(z)}{\psi'(z)}-2\Big\{\frac{1}{\psi'(t_r)}+\frac{1}{\psi'(t_{r+1})}+\frac{1}{\psi'(t_{r+2})}+\ldots\text{to infinity}\Big\}$$

$$-\int_0^z\frac{d}{dt}\Big(\frac{1}{\psi'}\Big)\ |\sin\psi|\ dt. \qquad \text{(A)}$$

But

$$\psi(z)\succ\log\frac{1}{z^2}.$$

Therefore

$$\psi'(z)\succ\frac{1}{z},$$

and, consequently,

$$\frac{1}{\psi'(z)}\prec z.$$

Hence the parts, contributed to $\frac{F(z)}{z}$ by the first and third parts of (A), both tend to 0 with z. Therefore

$$F'(0)=\operatorname*{Lim}_{z=0}\frac{F(z)}{z}=-2\operatorname*{Lim}_{z=0}\frac{1}{z}\Big\{\frac{1}{\psi'(t_r)}\pm\frac{1}{\psi'(t_{r+1})}+\ldots\text{ to infinity}\Big\}.$$

(a) *Case*: $\psi(t)=\frac{1}{t}$.

According to a result due to Abel*,

the series

$$P(r) \equiv \phi(r)+\phi(r+1)+\phi(r+2)+\ldots \quad \text{to infinity}$$

$$=\int_r^{\infty}\phi(t)dt+\tfrac{1}{2}\phi(r)-2\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\cdot\frac{\phi(r+it)-\phi(r-it)}{2i}dt,$$

if $\phi(x)$ and $\int\phi(x)dx$ vanish for $x=\infty$.

Now, in the present case,

$$\phi(r)=\frac{1}{\psi'(t_r)}=-\frac{1}{\pi^2}\cdot\frac{1}{(r+\frac{1}{2})^2}.$$

Therefore the conditions of Abel's theorem are satisfied. Further,

$$\frac{1}{2i}\left\{\frac{1}{(r+\frac{1}{2}+it)^2}-\frac{1}{(r+\frac{1}{2}-it)^2}\right\}=\frac{-1}{(r+\frac{1}{2})^2+t^2}\sin\theta,$$

where θ is a real angle. Therefore

$$\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\cdot\frac{\phi(r+it)-\phi(r-it)}{2i}dt=\frac{K}{(r+\frac{1}{2})^2},$$

where K is numerically less than a finite quantity independent of r.

Thus

$$P(r)=-\frac{1}{\pi^2}\left\{\frac{1}{(r+\frac{1}{2})}+\frac{K_1}{(r+\frac{1}{2})^2}\right\},$$

where K_1 is numerically less than a finite quantity independent of r.

Therefore

$$F'(0)=\frac{2}{\pi^2}\operatorname*{Lim}_{z=0}\frac{1}{z}\left\{\frac{1}{r+\frac{1}{2}}+\frac{K_1}{(r+\frac{1}{2})^2}\right\}.$$

* See his paper, "L'integrale finie $\Sigma^n\phi(x)$ exprimée par une integrale definie simple" (*Oeuvres*, t. 1, pp. 34-39, specially p. 38).

But z lies between t_r and t_{r-1}. Therefore

$$\frac{1}{z}=(r+\tfrac{1}{2}-k)\pi,$$

where $$0 \leq k < 1.$$

Therefore*

$$F'(0)=\frac{2}{\pi}.$$

(b) *Case*: $$\psi(t)=\frac{1}{t^m}.$$

Proceeding as in the case in (a), we find

$$t_r=\frac{1}{\pi^{\frac{1}{m}}(r+\frac{1}{2})^{\frac{1}{m}}},\quad \phi(r)=-\frac{1}{m\pi^{\frac{m+1}{m}}}\cdot\frac{1}{(r+\frac{1}{2})^{\frac{m+1}{m}}},$$

$$P(r)=-\frac{1}{\pi^{\frac{m+1}{m}}}\left\{\frac{1}{(r+\frac{1}{2})^{\frac{1}{m}}}+\frac{K_2}{(r+\frac{1}{2})^{\frac{m+1}{m}}}\right\},$$

* This result can be proved as follows without using Abel's theorem :—

The sum of the series

$$\frac{1}{(r+\frac{1}{2})^2}+\frac{1}{(r+\frac{3}{2})^2}+\ldots \text{ to infinity}$$

lies between $\frac{1}{r-\frac{1}{2}}$ and $\frac{1}{r+\frac{1}{2}}$, as the general term lies between

$$\frac{1}{\left(r+\frac{2n-3}{2}\right)\left(r+\frac{2n-1}{2}\right)} \text{ and } \frac{1}{\left(r+\frac{2n-1}{2}\right)\left(r+\frac{2n+1}{2}\right)}.$$

Therefore $P(r)$ behaves as $-\frac{1}{\pi^2}\cdot\frac{1}{r}$ as r tends to infinity.

nce $\frac{F(z)}{z}$ tends to $\frac{2}{\pi}$.

where K_2 is numerically less than a finite quantity independent of r.

Thus $$F'(0)=\frac{2}{\pi}.$$

(c) *Case*: $$\psi(t)=\left(\log\frac{1}{t}\right)^{1+k},\quad k>1.$$

Proceeding as in the preceding cases, we find

$$t_r=e^{-\{(r+\frac{1}{2})\pi\}^{\frac{1}{1+k}}},\qquad \phi(r)=-\frac{1}{1+k}\,\frac{e^{-\{(r+\frac{1}{2})\pi\}^{\frac{1}{1+k}}}}{\{(r+\frac{1}{2})\pi\}^{\frac{k}{1+k}}},$$

$$P(r)=-\frac{1}{\pi}e^{-\{(r+\frac{1}{2})\pi\}^{\frac{1}{1+k}}}\left[1+\frac{K_3}{\{(r+\frac{1}{2})\pi\}^{\frac{k}{1+k}}}\right],$$

where K_3 is numerically less than a finite quantity independent of r.

Thus $$F'(0)=\frac{2}{\pi}.$$

(d) *Case*: $$\psi(t)=e^{\frac{m}{t^n}},\qquad m>0,\quad n>0.$$

Proceeding as in the preceding cases, we find

$$t_r=\frac{m^{\frac{1}{n}}}{\left\{\log\left(r\pi+\frac{\pi}{2}\right)\right\}^{\frac{1}{n}}},$$

$$\phi(r)=-\frac{m^{\frac{n+1}{n}}}{mn}\;\frac{1}{(r+\frac{1}{2})\pi\left\{\log\left(r\pi+\frac{\pi}{2}\right)\right\}^{\frac{n+1}{n}}},$$

$$P(r)=-\frac{m^{\frac{n+1}{n}}}{m\pi}\left[\frac{1}{\left\{\log\left(r\pi+\frac{\pi}{2}\right)\right\}^{\frac{1}{n}}}\left\{1+\frac{K_4}{(r+\frac{1}{2})\pi}\right\}\right],$$

where K_4 is numerically less than a finite quantity independent of r.

Thus $$F'(0)=\frac{2}{\pi}.$$

Generally, it can be shown that, when $\psi(t) \succeq \log \frac{1}{t^2}$, $F'(0)$ exists but equals a quantity different from 0.

§ 2.

Case: $f(x_0+2t)+f(x_0-2t)=\chi(t)\cos\psi(t)$.

When $F'_1(0)$ exists, $F'(0)$ exists and is different from zero.

Proof:—

It has been proved by me in a paper, to be published soon, that

$F'_1(0)$ exists only when

$$\psi(t) \succ \log\frac{1}{t^2} \text{ and } \frac{\chi(t)}{\psi'(t)} \prec t.$$

It is obvious that $F'(0)$ is non-existent when $F'_1(0)$ is non-existent. Therefore we need consider for the purpose of the proof only the case in which the conditions given above are satisfied by χ and ψ.

Assuming for the sake of simplicity and fixity of ideas, that $\chi(t)$ is positive, we have, by a procedure similar to that used in the beginning of §1,

$$F(z)\equiv\int_0^z \chi(t)\,|\cos\psi(t)|\,dt$$

$$=(-1)^r\,\frac{\chi(z)}{\psi'(z)}\sin\psi(z)$$

$$-2\left\{\frac{\chi(t_r)}{\psi'(t_r)}+\frac{\chi(t_{r+1})}{\psi'(t_{r+1})}+\frac{\chi(t_{r+2})}{\psi'(t_{r+2})}+\dots\text{ to infinity}\right\}$$

$$-\int_0^z \frac{d}{dt}\left(\frac{\chi}{\psi'}\right)|\sin\psi|\,dt, \qquad \dots \text{(B)}$$

But $$\frac{\chi(z)}{\psi'(z)} \prec z.$$

Hence the parts, contributed to $\frac{F(s)}{s}$ by the first and third parts of (B), both tend to 0 with s. Therefore

$$F'(0) = \operatorname*{Lim}_{s=0} \frac{F(s)}{s} = -2 \operatorname*{Lim}_{s=0} \frac{1}{s} \left\{ \frac{\chi(t_r)}{\psi'(t_r)} + \frac{\chi(t_{r+1})}{\psi'(t_{r+1})} + \frac{\chi(t_{r+2})}{\psi'(t_{r+2})} \right.$$
$$\left. + \ldots \text{ to infinity} \right\}.$$

(*a*) *Case*: $\psi(t) = \frac{1}{t^n}$, $\chi(t) = \frac{1}{t^m}$, $n > m > 0$, $m < 1$. Then

$$t_r = \frac{1}{\pi^{\frac{1}{n}} (r+\frac{1}{2})^{\frac{1}{n}}},$$

and
$$\frac{\chi(t_r)}{\psi'(t_r)} = -\frac{1}{n} \frac{1}{\{\pi(r+\frac{1}{2})\}^{\frac{n+1-m}{n}}}.$$

Thus in Abel's theorem,

$$\phi(r) = -\frac{1}{n\pi^{\frac{n+1-m}{n}}} \frac{1}{(r+\frac{1}{2})^{\frac{n+1-m}{n}}}.$$

Therefore $\phi(x)$ and $\int\phi(x)dx$ vanish for $x=\infty$ and Abel's theorem is applicable to the present case.

Therefore $F(s)$ behaves as

$$\int_r^\infty \phi(t)dt,$$

i.e. as
$$-\frac{1}{(1-m)\pi^{\frac{n+1-m}{n}}} \frac{1}{(r+\frac{1}{2})^{\frac{1-m}{n}}},$$

as r tends to infinity and, consequently, s tends to 0.

Therefore $\frac{F(s)}{s}$ behaves as $(r+\frac{1}{2})^{\frac{m}{n}}$.

Therefore $F'(0) = \infty$.

(b) *Case*: $\psi(t) = \left(\log \frac{1}{t}\right)^n, \quad \chi(t) = \left(\log \frac{1}{t}\right)^m, \; m+1 < n.$

Then

$$t_r = e^{-\{\pi(r+\frac{1}{2})\}^{\frac{1}{n}}}, \quad \phi(r) = -\frac{1}{n} e^{-\{\pi(r+\frac{1}{2})\}^{\frac{1}{n}}} \{\pi(r+\tfrac{1}{2})\}^{\frac{m+1-n}{n}}.$$

And Abel's theorem is applicable, since $\phi(x)$ vanishes for $x = \infty$ and

$$\int \phi(x)dx \text{ behaves as } e^{-x^{\frac{1}{n}}} x^{\frac{m}{n}}$$

for x tending to infinity. Therefore

$$\frac{F(z)}{z} \text{ behaves as } \frac{e^{-r^{\frac{1}{n}}} r^{\frac{m}{n}}}{e^{-r^{\frac{1}{n}}}}, \text{ i.e., as } r^{\frac{m}{n}}$$

when r tends to infinity, and consequently, z tends to 0. Therefore

$$F'(0) = \infty.$$

Generally, it can be shown that, when $\frac{\chi}{\psi} \sim t$, $F'(0)$ exists and is different from zero.

§ 3.

Lebesgue's Criterion.

In order to see if Lebesgue's criterion can enable us to decide whether the Fourier series of a given function $f(x)$ is summable (C1) at a given point x_0, the criterion may be stated as follows*:—

If S is a quantity such that

$$\int_0^z \left| f(x_0+2t) + f(x_0-2t) - 2S \right| dt$$

* See Schlesinger and Plessner's book, *Lebesguesche Integrale und Fouriersche Reihen* (Berlin, 1926), p. 211.

has a differential co-efficient at $s=0$ *equal* to 0, then S is the limit to which Cesàro's mean $S_n(x_0)$ tends as n tends to infinity.

The form in which the criterion was originally* given by Lebesgue serves best the purpose of showing that the series is summable (C1) almost everywhere; but in that form the criterion is unsuitable for testing the summability at a given point x_0, where $f(x)$ has a discontinuity of the second kind so that really $f(x_0)$ has no definite meaning.

§4.

The Fourier series, if summable (C1), has sum zero.

(*a*) When

$$f(x_0+2t)+f(x_0-2t)=\cos\psi(t),$$

the series is summable (C1) if

$$\psi \succ \log\frac{1}{t^2};$$

and in that case the sum is zero.

This result, which was deduced† by me with the help of a result of

* According to Prof. Hobson (See his *Theory of Functions of a Real Variable*, Vol. 2, 2nd Edition, p. 561), " a more general theorem than that of" Fejér " has been obtained by Lebesgue. He has in fact shewn that the Cesàro sum converges to $f(x)$ at any point for which

$$\int_0^t \left|f(x+t)+f(x-t)-2f(x)\right| dt$$

has a differential coefficient equal to zero, at the point $t=0$."

In the *Encyklopädie der mathematischen Wissenschaften*, Bd. II C 10, p. 1206, the same form of the criterion has been given as by Hobson.

† This *Bulletin*, Vol. 18, pp. 156-158.

Professor W. H. Young, can also be deduced by using a criterion* given by Professor G. H. Hardy and Mr. J. E. Littlewood.

(b) When

$$f(x_0+2t)+f(x_0-2t)=\chi(t)\cos\psi(t),$$

the series is summable (C1) if

$$\chi \prec \frac{1}{t},\ \psi \succ \log\frac{1}{t^2},\ \frac{\chi}{\psi'} \prec t;$$

and in that case the sum is zero.

This result was also deduced † by me with the help of Young's result; the criterion of Hardy and Littlewood does not apply to this case.

§5.

Failure of Lebesgue's Criterion.

Lebesgue's criterion fails in every case in which

$$f(x_0+2t)+f(x_0-2t)=\cos\psi(t) \quad \text{or} \quad \chi(t)\cos\psi(t).$$

For, in such a case S is *always* zero as shown in §4; and, further,

$$\int_0^x \left| f(x_0+2t)+f(x_0-2t)-2S \right| dt$$

* See *Proc. L. M. S., Ser. 2*, Vol. 17, pp. XIII-XIV. The criterion is applicable to *bounded* functions only and is this:

$$\text{If} \quad \frac{1}{2h}\int_{x_0-h}^{x_0+h} f(t)dt \longrightarrow S$$

when $h \longrightarrow 0$, then S is the sum. In the present case, my result on the differentiability of the integral-function gives the limit of the right side to be 0; hence S is 0.

† This *Bulletin*, Vol. 18, pp. 183-184.

has *always* a differential coefficient at $z=0$ *different* from zero, as shown in §1 and §2.

The various proofs* of the criterion start with the assumption that

$$\int_0^z \left| f(x_0+2t)+f(x_0-2t)-2S \right| dt$$

has a differential coefficient *equal* to 0 at $z=0$, and then deduce from the assumption that S is the limit to which the mean $S_n(x_0)$ must tend with n tending to infinity. As no proof considers explicitly the possibility of $f(x_0+2t)+f(x_0-2t)$ having a discontinuity of the second kind, the assumption in reality amounts to the exclusion of the possibility of such a discontinuity. The proofs, thus, establish the validity of the criterion only for the case in which $\{f(x_0+2t)+f(x_0-2t)\}$ has a limit when t tends to 0.

* See Lebesgue's proof in *Math. Ann.*, Vol. 61, pp. 274-276 ; Hobson, *l.c.* ; Schlesinger and Plessner, *l.c.*

2

ON THE PROOFS OF SOME RESULTS GIVEN BY RAMANUJAN ABOUT THE COMPLEX MULTIPLICATION OF ELLIPTIC FUNCTIONS

BY

S. C. MITRA.

Introduction.

1. The object of the present paper is to supply the proofs of two results about the complex multiplication of elliptic functions, *viz.*, the values of the moduli for the determinants 217 and 141, which the late S. Ramanujan* gave without proof. The proofs are based on the method of R. Russel.†

2. If the modular equations of the mth and nth orders be written in the form

$$F_m(\Omega_1, \omega)=0,\ F_n(\Omega, \Omega_1)=0,$$

then

$$\Omega_1=\frac{\omega+16r}{m},\ \Omega=\frac{\Omega_1+16s}{n}.$$

Putting

$$\Omega=-\frac{1}{\omega}+2t$$

we get

$$\omega=mnt-8r-8ms$$
$$+\sqrt{-[mn-(mnt-8r-8ms)^2]} \qquad \dots \quad (A)$$

Modulus corresponding to $\omega=\sqrt{-217}$.

3. If in (A) we take

$$m=23,\ n=11,$$

* *Quarterly Journal of Mathematics*, Vol. XLV.

† *Proc. L. M. S.* Series 1, Vol. XXI.

then in order that

$$mnt-8r-8ms=6,$$

we must have

$$t=6.$$

The modular equation of the 11th order can be written in the form

$$-\sqrt{k'\lambda}+\sqrt{k\lambda'}=1+2\sqrt[6]{4k\lambda k'\lambda'}$$

Let

$$(4k\lambda k'\lambda')^{\frac{1}{12}}=x.$$

Then,

$$-\sqrt{k'\lambda}+\sqrt{k\lambda'}=1+2x^2.$$

or,
$$k'\lambda+k\lambda'=1+4x^2+4x^4+x^6. \qquad \ldots \text{(A}''\text{)}$$

The modular equation of the 23rd order is

$$\sqrt[4]{k\lambda}+\sqrt[4]{k'\lambda'}+\sqrt{2}\,\sqrt[12]{4k\lambda k'\lambda'}=1.$$

Therefore,

$$\sqrt[4]{k\lambda}+\sqrt[4]{k'\lambda'}=1-\sqrt{2}\,x$$

and

$$\sqrt{k\lambda}+\sqrt{k'\lambda'}=1-2\sqrt{2}x+2x^2-\sqrt{2}x^3.$$

Hence,

$$k\lambda+k'\lambda'=1-4\sqrt{2}x+12x^2-10\sqrt{2}x^3+12x^4-4\sqrt{2}x^5+x^6. \qquad \text{(B}''\text{)}$$

4. Squaring and adding (A″) and (B″), we have

$$(1-4\sqrt{2}x+12x^2-10\sqrt{2}x^3+12x^4-4\sqrt{2}x^5+x^6)^2$$
$$+(1+4x^2+4x^4+x^6)^2=1+x^{12}.$$

Putting

$$x+\frac{1}{x}=\sqrt{2}y,$$

the equation becomes

$$4y^6-32y^5+116y^4-152y^3+105y^2-36y+5=0. \qquad \ldots \quad (C'')$$

This is equivalent to

$$(2y^3-8y^2+9y-2)^2+(4y^2-1)^2=0.$$

or
$$(2y^3-8y^2+9y-2)+i(4y^2-1)=0.$$

or
$$\left(y-\frac{1+i}{2}\right)\{2y^2-(7-5i)y+(3-i)\}=0.$$

The quadratic factor corresponds to

$$\omega=6+\sqrt{-217}.$$

On solving the quadratic equation, we get

$$y=\frac{(7+\sqrt{31})-i(5+\sqrt{31})}{4}.$$

5. We have obtained

$$k'\lambda+k\lambda'=1+4x^2+4x^4+x^6.$$

From this we get

$$(k\lambda+k'\lambda')^2=-2x^2(2+3x^2+2x^4)^2$$

or
$$k\lambda+k'\lambda'=-\sqrt{2}ix\ (2+3x^2+2x^4).$$

Therefore,

$$(2kk'+2\lambda\lambda')$$

$$=-2\sqrt{2}i(2x+11x^3+22x^5+22x^7+11x^9+2x^{11})$$

and
$$\frac{1}{2kk'}+\frac{1}{2\lambda\lambda'}$$

$$=-2\sqrt{2}i\left(\frac{2}{x}+\frac{11}{x^3}+\frac{22}{x^5}+\frac{22}{x^7}+\frac{11}{x^9}+\frac{2}{x^{11}}\right).$$

Hence,

$$\left(2kk'+\frac{1}{2kk'}\right)+\left(2\lambda\lambda'+\frac{1}{2\lambda\lambda'}\right)$$

$$=-2\sqrt{2}i\left(x^6+\frac{1}{x^6}\right)\left\{2\left(x^5+\frac{1}{x^5}\right)+11\left(x^3+\frac{1}{x^3}\right)+22\left(x+\frac{1}{x}\right)\right\}.$$

Noticing that

$$x+\frac{1}{x}=\sqrt{2}y$$

where y has the value obtained above,

we get after some reduction,

$$\left(2kk'+\frac{1}{2kk'}\right)+\left(2\lambda\lambda'+\frac{1}{2\lambda\lambda'}\right)$$

$$=-\{(6236+1120\sqrt{31})^2+(17640+3168\sqrt{31})^2\}$$

$$=-(700066640+125735680\sqrt{31}).$$

Again

$$\left(2kk'+\frac{1}{2kk'}\right)\left(2\lambda\lambda'+\frac{1}{2\lambda\lambda'}\right)$$

$$=\left(x^{12}+\frac{1}{x^{12}}\right)+\left(\frac{kk'}{\lambda\lambda'}+\frac{\lambda\lambda'}{kk'}\right)$$

and $\left(\frac{kk'}{\lambda\lambda'}+\frac{\lambda\lambda'}{kk'}\right)+2$

$$=-8\left\{2\left(x^5+\frac{1}{x^5}\right)+11\left(x^3+\frac{1}{x^3}\right)+22\left(x+\frac{1}{x}\right)\right\}^2.$$

Therefore

$$\left(2kk'+\frac{1}{2kk'}\right)\left(2\lambda\lambda'+\frac{1}{2\lambda\lambda'}\right)$$

$$=2(6236+1120\sqrt{31})^2-2(17640+3168\sqrt{31})^2-4$$

$$=-(1089036900+195596800\sqrt{31}).$$

Therefore,

$$2kk'+\frac{1}{2kk'}$$

is a root of the equation

$$z^2+(700066540+125735680\sqrt{31})z$$

$$-(1089036900+195596800\sqrt{31})=0.$$

6. Now in this case

$$\sqrt[4]{k}=\phi(6+\sqrt{-217})$$

$$=e^{\frac{3\pi i}{4}}\phi(\sqrt{-217}).$$

If
$$y=\phi^{4}(\sqrt{-217})\psi^{4}(\sqrt{-217})$$
then $z=-y$.

The equation becomes

$$y^{3}-\{700066640+125735680\sqrt{31}\}y$$
$$-\{1089036900+195596800\sqrt{31}\}=0.$$

Solving, we have

$$y=2kk'+\frac{1}{2kk'}$$
$$=140\{2500238+449056\sqrt{31}\}$$
$$+10\sqrt{7}\{350066647847$$
$$+62873825297792\sqrt{31}\}^{\frac{1}{2}}$$

whence it is obvious that

$$(2kk')^{\frac{1}{4}}+\frac{1}{(2kk')^{\frac{1}{4}}}$$
$$=\frac{(17+3\sqrt{31})+\sqrt{7}\ (7+\sqrt{31})}{2}. \qquad \ldots \quad (1)$$

Ramanujan has given, without proof, the result in the form

$$G^{3}_{217}=\left\{\sqrt{\left(\frac{9+4\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{11+4\sqrt{7}}{2}\right)}\right\}$$
$$\times\left\{\sqrt{\left(\frac{12+5\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{16+5\sqrt{7}}{4}\right)}\right\},$$

where

$$G_{217}=\frac{1}{(2kk')^{\frac{1}{12}}}.$$

Forming $(G_{517})^2+\dfrac{1}{(G_{517})^2}$, we find it to equal the right side of (1)

Modulus corresponding to $\omega=\sqrt{-141}$.

7. In this case we combine the modular equation of the 3rd order with that of the 47th order.

The modular equation of the 3rd order is

$$\sqrt{k'\lambda}+\sqrt{k\lambda'}=1.$$

Putting

$$(4k\lambda k'\lambda')^{\frac{1}{12}}=x$$

we have

$$k'\lambda+k\lambda'=1-x^6.$$

$$(k\lambda+k'\lambda')^2=1+4k\lambda k'\lambda'-(k'\lambda+k\lambda')^2=2x^6.$$

or, $$k\lambda+k'\lambda'=\sqrt{2}x^3.$$

or, $$\sqrt{k\lambda}+\sqrt{k'\lambda'}=(\sqrt{2}x^3+x^6)^{\frac{1}{2}}.$$

or, $$\sqrt[4]{k\lambda}+\sqrt[4]{k'\lambda'}=\{(\sqrt{2}x^3+x^6)^{\frac{1}{2}}+\sqrt{2}x^3\}^{\frac{1}{2}}.$$

The modular equation of the 47th order is

$$P^2-4Q-P(4R)^{\frac{1}{3}}-2(4R)^{\frac{2}{3}}=0$$

where

$$P=\{(\sqrt{2}x^3+x^6)^{\frac{1}{2}}+\sqrt{2}x^3\}^{\frac{1}{2}}+1$$

$$Q=\frac{x^3}{\sqrt{2}}+\{(\sqrt{2}x^3+x^6)^{\frac{1}{2}}+\sqrt{2}x^3\}^{\frac{1}{2}}$$

$$R=\frac{x^3}{\sqrt{2}}.$$

Substituting in the above modular equation, we have

$$(\sqrt{2}x^3+x^6)^{\frac{1}{2}}-(\sqrt{2}x^3+4x^2+\sqrt{2}x-1)$$

$$=(2+\sqrt{2}x)\{(\sqrt{2}x^3+x^6)^{\frac{1}{2}}+\sqrt{2}x^3\}^{\frac{1}{2}}.$$

After some reduction, we get

$$(3x^6+6\sqrt{2}x^5+12x^4+3\sqrt{2}x^3-6x^2-2\sqrt{2}x+1)^2$$

$$=(x^6+\sqrt{2}x^3)\{8x^6+40\sqrt{2}x^5+148x^4+128\sqrt{2}x^3$$

$$+112x^2+24\sqrt{2}x+4\},$$

or,
$$(x^{12}+1)-4\sqrt{2}(x^{11}+x)-4(x^{10}+x^2)+26\sqrt{2}(x^9+x^3)$$

$$-12(x^8+x^4)-184\sqrt{2}(x^7+x^5)-428x^6=0.$$

Putting

$$x+\frac{1}{x}=\sqrt{2}y,$$

we have

$$4y^6-16y^5-20y^4+92y^3+13y^2-282y-207=0,$$

which breaks up into the factors

$$(2y^2-6y-9)(2y^4-2y^3-7y^2+16y+23)=0.$$

The quadratic factor corresponds to $\omega=\sqrt{-141}$.

One root of the quadratic equation is

$$y=\frac{3+3\sqrt{3}}{2}.$$

Therefore,

$$x+\frac{1}{x}=\frac{3+3\sqrt{3}}{\sqrt{2}}.$$

or,
$$x=\frac{(3+3\sqrt{3})}{2\sqrt{2}}\pm\frac{(14+9\sqrt{3})^{\frac{1}{2}}}{2}.$$

$\sqrt{3})^{\frac{1}{3}}(17+9\sqrt{3})$

$\frac{1}{x^{3}}-x^{3}\Big)$

$'3)(14+9\sqrt{3})^{\frac{1}{3}}.$

$^{12}(3491+2016\sqrt{3})$

$491+2016\sqrt{3})^{\frac{1}{3}}$

$+9\sqrt{3})(14+9\sqrt{3})^{\frac{1}{2}}$

$-3.(3491+2016\sqrt{3})^{\frac{1}{3}}\}.$ .... (1)

hout proof, in the form

$\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{\frac{1}{3}}$

$\frac{9\sqrt{3}}{}\Big)+\sqrt{\left(\frac{14+9\sqrt{3}}{4}\right)}\Big\}$, ... (2)

$\frac{1}{kk'}\Big)^{\frac{1}{12}}.$

ocal of the right side of (2) equals

)r. G. Prasad, D.Sc., who kindly
ok great interest in the preparation

, No. 1.

JCU 40

On a generalization of Lagrange's inversion formula

BY

Nripendranath Ghosh

(*Calcutta University*)

1. The object of this paper is simply to generalize the well-known inversion formula of Lagrange. The method adopted is the same as that used by Hermite in obtaining Lagrange's formula. The importance of the present formula lies in the readiness with which it can be applied to solve equations, algebraic or transcendental, by means of infinite series. It is believed that this generalization is new. A generalisation, arrived at by a different process, has been given by M. Kössler in the *Proceedings of the London Mathematical Society*, series 2, Vol. 20. He has built up step by step the general formula proposed by him.

2. Let $f(z)$ and $\phi(z)$ be two functions analytic in a certain region D containing a point a. Let C be a circle with the centre a and radius r lying entirely in the region D, and such that within it the equation $\phi(z)-\beta=0$ has a single root $z=a$. Suppose that a variable parameter α is so determined that along the entire circumference of the circle C, $|\alpha f(z)| < |\phi(z)-\beta|$. Then the equation $F(z)=\phi(z)-\beta-\alpha f(z)=0$ has only one root within the curve C.* Let ξ denote this particular root and let $\pi(z)$ be an analytic function in this circle C.

The function $\dfrac{\pi(z)}{F(z)}$ has now a single pole in the interior of C, at the point $z=\xi$ and the corresponding residue is $\dfrac{\pi(\xi)}{F'(\xi)}$. We, then, have

$$\frac{\pi(\xi)}{F'(\xi)}=\frac{1}{2\pi i}\int_{(c)}\frac{\pi(z)}{F(z)}\,dz=\frac{1}{2\pi i}\int_{(c)}\frac{\pi(z)\,dz}{\phi(z)-\phi(a)-\alpha f(z)},$$

where we replace β by $\phi(a)$.

* Goursat "*Mathematical Analysis*," Vol. II, Part 1. Art. 49, Note.

Developing the integral on the right in powers of a, we find

$$\frac{\pi(\xi)}{F'(\xi)} = H_0 + aH_1 + a^2H_2 + \dots\dots\dots + a^nH_n + R_{n+1},$$

where

$$H_0 = \frac{1}{2\pi i}\int_{(C)} \frac{\pi(z)dz}{\phi(z)-\phi(a)},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$H_n = \frac{1}{2\pi i}\int_{(C)} \frac{\pi(z)\{f(z)\}^n}{\{\phi(z)-\phi(a)\}^{n+1}},$$

$$R_{n+1} = \frac{1}{2\pi i}\int_{(C)} \frac{\pi(z)}{\phi(z)-\phi(a)-af(z)} \cdot \left[\frac{af(z)}{\phi(z)-\phi(a)}\right]^{n+1} dz.$$

The integrals H_0, H_1, $H_2 \dots\dots H_n$ being evaluated can be expressed in the form

$$H_0 = \frac{\pi(a)}{\phi'(a)},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$H_n = \frac{1}{n!}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^n \frac{\pi(a)\{f(a)\}^n}{\phi'(a)}.$$

The integral R_{n+1} approaches zero as n increases indefinitely. Hence we obtain the following development in series:

$$\frac{\pi(\xi)}{F'(\xi)} = \frac{\pi(a)}{\phi'(a)} + \sum_{n=1}^{\infty} \frac{a^n}{n!}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^n \frac{\pi(a)\{f(a)\}^n}{\phi'(a)}.$$

If we take $\pi(z)=\psi(z)\{\phi'(z)-af'(z)\}$, where $\psi(z)$ is an analytic function in the circle C, then the above gives the development of $\psi(\xi)$ in a convergent series

$$\psi(\xi) = \psi(a) + \sum_{n=1}^{\infty} \frac{a^n}{n!}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^{n-1} \frac{\{f(a)\}^n\psi'(a)}{\phi'(a)} \quad \dots \quad \text{(I)}$$

which is my *generalization of Lagrange's formula.*

If $z=a$ happens to be a double root of the equation $\phi(z)-\beta=0$, then $\phi'(a)=0$ and the above formula ceases to be applicable.

3. This special case requires a separate treatment. We observe that the equation $F(z)=\phi(z)-\beta-af(z)=0$ has now a pair of roots within the curve C. Let ξ_1 and ξ_2 denote these roots and let $\pi(z)$, as before, be an analytic function in the circle C. We have then

$$\frac{\pi(\xi_1)}{F'(\xi_1)}+\frac{\pi(\xi_2)}{F'(\xi_2)}=\frac{1}{2\pi i}\int_{(C)}\frac{\pi(z)dz}{F(z)}\text{, which, as before, becomes}$$

$$H_0+aH_1+\ldots\ldots+a^nH_n+R_{n+1}.$$

The integrals H_0, H_1,..... ..H_n are now to be evaluated as follows :

As $\phi'(a)=0$, we may write

$\phi(z)-\phi(a)=(z-a)^2\,\phi_1(z)$, where $\phi_1(z)$ denotes

$$\frac{\phi''(a)}{2!}+(z-a)\frac{\phi'''(a)}{3!}+(z-a)^2\frac{\phi^{iv}(a)}{4!}+\ldots\ldots\ldots$$

Hence $H_0=\dfrac{d}{dz}\left\{\dfrac{\pi(z)}{\phi_1(z)}\right\}$,

$$H_1=\frac{1}{3!}\frac{d^3}{dz^3}\left\{\frac{\pi(z)f(z)}{\{\phi_1(z)\}^2}\right\},$$

... .

$$H_n=\frac{1}{(2n+1)!}\frac{d^{2n+1}}{dz^{2n+1}}\left(\frac{\pi(z)\{f(z)\}^n}{\{\phi_1(z)\}^{n+1}}\right),$$

where in each we replace z by a after performing the indicated differentiations.

The integral R_{n+1} approaches zero as before.

Putting then $\pi(z)=\psi(z)\{\phi'(z)-af'(z)\}$ we obtain the development of $\psi(\xi_1)+\psi(\xi_2)$ in a convergent series of the form

$$P_0+P_1a+P_2a^2+\ldots\ldots\ldots+P_na^n+\ldots \quad \ldots. \quad \ldots \quad \text{(II)}$$

where

$$P_0 = \frac{d}{dz}\left(\frac{\psi(z)\phi'(z)}{\phi_1(z)}\right), \; z=a,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$P_n = \frac{1}{(2n+1)!}\frac{d^{2n+1}}{dz^{2n+1}}\left(\frac{\psi(z)\phi'(z)\{f(z)\}^n}{\{\phi_1(z)\}^{n+1}}\right)$$

$$-\frac{1}{(2n-1)!}\frac{d^{2n-1}}{dz^{2n-1}}\left(\frac{\psi(z)f'(z)\{f(z)\}^{n-1}}{\{\phi_1(z)\}^n}\right), \; z=$$

The case when $z=a$ is a multiple root of the equation ϕ may be treated in the same manner.

4. The application of the above formulae to the solution tions is obvious. Given any equation, we can express it in th

$$F(z)=\phi(z)-\beta-af(z)=0$$

in a variety of ways, and in a particular example it is not di choose the suitable forms in conformity with the conditions o gence indicated above. In the actual numerical computation the mere knowledge of convergence is, however, not sufficient contrivance is needed to obtain rapidly convergent series.

My best thanks are due to Prof. Ganesh Prasad for his kin and encouragement.

ON THE FAILURE OF LEBESGUE'S CRITERION FOR THE SUMMABILITY (C2) OF THE FOURIER SERIES OF A FUNCTION AT A POINT WHERE THE FUNCTION HAS A CERTAIN TYPE OF DISCONTINUITY OF THE SECOND KIND

BY

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The object of the present paper is to formulate a definite answer to the question: Does Professor Lebesgue's criterion* for the summability (C2) of the Fourier series of a function $f(x)$ enable us to settle the question of summability at a point x_0 in *every* case in which $f(x)$ has a discontinuity of the second kind at x_0? I show that the *answer is in the negative*; by giving a class of functions each of which has a certain type of infinite discontinuity of the second kind, and for each of which the series is summable (C2) although Lebesgue's criterion is not satisfied.

It is believed that the aforesaid question has not been considered by any previous writer.

Throughout this paper,

$$S \equiv \text{the sum } (C\ 2),$$

$$\phi(t) \equiv f(x_0+2t)+f(x_0-2t)-2S,$$

$$\Phi(t) \equiv \int_0^t \phi(t_1)dt_1.$$

It may be noted that, Lebesgue's criterion for the summability (C 2) to S at x_0 is that $\Phi'(0)$ should be zero.

* *Math. Ann.*, Bd. 61, 1905, pp. 278-279.

1. It is well-known that Fejér's mean

$$S_n(x_0) \equiv \frac{s_0+s_1+\dots+s_{n-1}}{n},$$

where

$$s_n(x) \equiv \tfrac{1}{2}a_0 + \sum_1^n (a_r \cos rx + b_r \sin rx),$$

behaves for large values of n as

$$\frac{1}{n\pi}\int_0^a \left(\frac{\sin nt}{t}\right)^2 \{f(x_0+2t)+f(x_0-2t)\}dt,$$

where $a > 0$ is independent of n but as small as we choose.

Therefore

$$\sigma_n(x_0) \equiv S_n(x_0) - S$$

behaves as

$$\frac{1}{n\pi}\int_0^a \left(\frac{\sin nt}{t}\right)^2 \phi(t)dt.$$

But, integrating by parts,

$$\int_0^a \left(\frac{\sin nt}{t}\right)^2 \phi(t)dt = \left(\frac{\sin na}{a}\right)^2 . \Phi(a)$$

$$+2\int_0^a \frac{\Phi(t)}{t} . \left(\frac{\sin nt}{t}\right)^2 dt - n\int_0^a \frac{\Phi(t)}{t} . \frac{\sin 2nt}{t} dt.$$

Assume that

$$\Phi(t) = t\chi(t) \cos \psi(t),$$

where χ and ψ are monotone and unbounded in the neighbourhood of $t=0$ and

$$\chi(t) \prec \frac{1}{t}.$$

Thus, neglecting terms which vanish with $\frac{1}{n}$,

$$\sigma_n(x_0) = \frac{1}{n\pi}\left(\frac{\sin n\alpha}{\alpha}\right)^2 \Phi(\alpha)$$

$$+ \frac{2}{n\pi}\int_0^\alpha \chi(t) \cos \psi(t) . \left(\frac{\sin nt}{t}\right)^2 dt$$

$$-\frac{1}{\pi}\int_0^\alpha \chi(t) \cos \psi(t) . \frac{\sin 2nt}{t} dt, \qquad \ldots \text{(A)}$$

2. Consider

$$\sum_n = \frac{\sigma_1 + \ldots + \sigma_n}{n},$$

and let the parts of $\sum_n$ due to the three terms of (A) be respectively $\sum_n'$, $\sum_n''$ and $\sum_n'''$. Then, obviously $\sum_n'$ tends to 0 with $\frac{1}{n}$, as

$$\frac{1}{n\pi}\left(\frac{\sin n\alpha}{\alpha}\right)^2 \Phi(\alpha) \text{ tends to } 0;$$

also* $\sum_n''$ tends to 0 with $\frac{1}{n}$, if $S=0$ and

$$\frac{\chi}{\psi'} \prec t. \qquad \ldots \text{(I)}$$

* See my paper, "On the summability (C 1) of the Legendre series of a function at a point where the function has a discontinuity of the second kind." (This *Bulletin*, Vol. 18, pp. 183-184). I take this opportunity to state that on page 179 of that paper an error has crept in which will be corrected later in another paper.

Thus the behaviour of $\sum_n$ depends on that of $\sum_n''$. But

$\sum_n'''$ behaves as

$$-\frac{1}{n\pi}\int_0^a \chi(t)\cos\psi(t).\ \frac{\sin 2t+\sin 4t+\cdots+\sin 2nt}{t}\,dt,$$

i.e., as
$$-\frac{1}{n\pi}\int_0^a \chi(t)\cos\psi(t).\ \frac{\sin(n+1)t\sin nt}{t\sin t}\,dt,$$

i.e., as
$$-\frac{1}{n\pi}\int_0^a \chi(t)\cos\psi(t).\ \left(\frac{\sin nt}{t}\right)^2 dt,$$

which tends to 0 if (I) is satisfied.

Therefore it is proved that, even if $\Phi'(0)$ is *not* existent, the Fourier series of $f(x)$ is summable (C 2) to 0 at x_0, provided that

$$\Phi(t)=t\chi(t)\cos\psi(t)$$

and
$$\frac{\chi}{\psi} \prec t.$$

Examples :

(1) $\chi=\left(\log\frac{1}{t^2}\right)^{\frac{1}{3}},\quad \psi=\left(\log\frac{1}{t^2}\right)^{\frac{2}{3}}.$

(2) $\chi=\left(\log\frac{1}{t^2}\right)^{m},\quad \psi=\left(\log\frac{1}{t^2}\right)^{1+k},$

where $m>0,\quad k>m.$

(3) $\chi=\frac{1}{t^m},\quad \psi=\frac{1}{t^k},\quad$ where $m>0,\quad k>m,\ m+k<1.$

PARALLAX IN HINDU ASTRONOMY.

BY

SUKUMAR RANJAN DAS

(Read on the 28th September, 1927.)

"At the change of the Moon," says Bhāskara in his *Siddhānta Śiromaṇi*, "it often so happens that an observer, placed at the centre of the earth, would find the sun, when far from the zenith, obscured by the intervening body of the moon; whilst another observer on the surface of the earth will not, at the same time, find him to be so obscured as the Moon will appear to him to be depressed from the line of vision extending from his eye to the Sun. Hence arises the necessity for the correction of parallax in celestial longitude and latitude in solar eclipses, in consequence of the difference of the distance of the sun and the moon. When the sun and the moon are in opposition, the earth's shadow envelopes the moon in darkness, its eclipse is seen equally by everyone on the earth's surface, and as the earth's shadow and the moon which enters it are at the same distance from the earth, there is, therefore, no call for the correction of the parallax in a lunar eclipse."*

The above extracts clearly indicate what parallax in longitude and latitude means and explains its importance in calculating eclipses. Hence the Hindu astronomers who were skilful in calculating eclipses must have been struck with this phenomenon for the solution of which they engaged their attention from the very early times. It is, therefore, no wonder why Bhāskara included it in the list of the principal problems to be tackled with in a comprehensive study of the astronomical science.†

It is strange that most of the Hindu astronomers made no mention of this phenomenon. Āryabhaṭa who lived in the fifth century A.D. did not pay any attention to it, nor did Lalla (sixth century A.D.) and Brahmagupta (born 598 A.D.) deal with this problem. But it is not true, as Mr. G. R. Kaye puts it, that in the works of the earlier

* *Siddhānta Śiromaṇi*, B. D. Sastri's edition, Golādhyāya, Chap. VIII, verses 3 and 4.

† *Siddhānta Śiromaṇi*, B.D. Sastri's edition, Golādhyāya, Chap II, verse 9.

period of astronomy there is no reference to parallax.* Varāhamihira, a contemporary of Lalla and almost of Āryabhaṭa, dealt with parallax at a considerable length in his *Pañcha-siddhāntikā* under the section dealing with old Sūrya Siddhānta. Now the old Sūrya Siddhānta was extant from very early times, probably from the beginning of the Christian Era. Varāhamihira only compiled that text of old Sūrya Siddhānta in his *Pañcha-siddhāntikā* with no modification of the main principles. No doubt, in the later Sūrya Siddhānta the phenomenon of parallax was dealt with more thoroughly and exhaustively, but the main principles were the same in both. The procedures in the two works will be described later on by way of comparison. In our opinion, at the time of Varāhamihira the problem of parallax was thoroughly solved.

It is known from modern Astronomy that the effect of parallax on a heavenly body is to depress it in the heavens. Bhāskara also says, "As the observer situated on the surface of the earth (elevated above the centre of the earth by half its diameter), sees the body depressed from its place (as found by a calculation made for the centre of the earth). Hence the parallax in longitude is calculated from the radius of the earth, as is also the parallax in latitude."† This couplet explains the effect of parallax and that it is the angle subtended at the body by the radius of the earth. Bhāskara has fully discussed the cause of parallax and the depression therefrom in the Golādhyāya of his *Siddhānta Śiromaṇi*.

Draw upon a smooth wall, the sphere of the earth, and the orbits of the moon and the sun on the same scale at proportionate distances; next draw a transverse diameter and also a perpendicular diameter to both orbits. Here the sphere of the earth and the orbits of the moon and sun are concentric. Let the orbits be cut by a horizontal transverse diameter at points on the horizon. The point above cut by the perpendicular diameter will represent the observer's zenith. Draw one line passing through the centre of the earth to the sun's disc, and another from the observer on the earth's surface to the sun's disc. The minutes contained in the arc, intercepted between these lines give the moon's parallax from the sun. [At the new moon, the sun and the moon will always appear by a line drawn from the centre of the earth to be in exactly the same place and to have the same longitude: but when the moon is observed from the surface of the earth in the *drik sutra* or line of vision, it appears to be depressed, and hence the name

* G. R. Kaye, *Ancient Hindu Spherical Astronomy*, J.A.S.B., 1919, p. 127.

† *Siddhānta Śiromaṇi*, Golādhyāya, Chap. VIII, verse 11.

Lambana or depression for parallax. When the new moon happens in the zenith, the line drawn from the earth's centre will coincide with that from its surface, hence a planet has no parallax when in that zenith.]*

† Let E be the centre of the earth; O, an observer on the surface of the earth; ZX and Z′X′ be the vertical circles passing through the Moon, M, and the Sun, S; X and X′ be the points of the horizon cut by the vertical circles ZX and Z′X′. Let Z be the zenith in the Moon's orbit and Z′ in that of the Sun. Now let EMS be a line drawn from the centre of the Earth to the Sun in which the Moon lies at the time of conjunction and let OS be the line of vision drawn from the observer to the Sun. The distance at which the Moon appears depressed from the line of vision in the vertical circles is the Moon's parallax from the Sun.

When the Sun reaches the zenith Z′, it is evident that the Moon will also then reach Z, and the line of vision of the observer and the line drawn from the centre of the earth will then coincide. There will, therefore, be no parallax in the zenith, *i.e.*, when the Sun is in the meridian.

Thus the parallax of the Moon from the Sun is equal to the difference between the parallaxes of the Sun and the Moon, separately found in the vertical circle. The parallax of the Moon from the Sun is arc $MM' = \angle M'OM$, the parallax of the Sun is $\angle OSM$ and the parallax of the moon is $\angle OME$. Hence the difference is $\angle M'OS = \angle M'OM$. This parallax which arises from the zenith distance of the planet, is called the common parallax or the parallax in altitude.‡

Let O be an observer on the surface of the earth; Z, the zenith; ZS, the vertical circle passing through the planet S; let a circle Z′mr be described with centre O and radius ES cutting OZ and OS produced at Z′ and r. Let a line sm be drawn parallel to OZ, then the arc Z′m will be equal to the arc ZS. Now, the planet S seen from E has a zenith distance ZS and seen from O, a zenith distance Z′r which is greater than ZS or Z′m; hence the apparent place r of the planet is depressed by mr in the vertical circle. This arc mr is, therefore, the common parallax of the planet. This is found as follows:—

Draw mn $\perp^r$ to Or and rt to OZ and let $P = ES$ or Or.

$h = EO$ or ZZ' or ms, $p = mr$, the common parallax.

* *Siddhānta Śiromaṇi*, Golādhyāya, B.D. Sastri's edition, verses 12-17.

† The diagrams used in this paper are ordinary figures of Spherical Astronomy. Readers are requested to draw them while going through this paper.

‡ See notes by Bapudev Sastri under verse 12 of Golādhyāya, and also *Vāsanā Bhāṣya* of Bhāskara in the chapter on eclipses.

d=ZS or Z'm, the true zenith distance of the planet;

and $\therefore$ $d+p$=Z'r, the apparent zenith distance of the planet

Then, mn=sin p and rt=sin $(d+p)$.

From $\triangle$'Ort and Smn, Or : rt=Sm : mn.

or $R:\sin(d+p)=h:\sin p$.

$$\therefore \quad \sin p=\frac{h\sin(d+p)}{R}.$$

Hence it is evident that when $\sin(d+p)=R$ or $d+p=90°$, the parallax will be greatest. Let it be denoted by P.

$$\therefore \quad \text{Sin P}=h$$

and $$\therefore \quad \sin p=\frac{\sin P\sin(d+p)}{R},$$

or, $$p=\frac{\rho\times\sin(d+p)}{R}, \text{ as } \sin p=p \text{ and } \sin P=P \text{ approximately.}$$

$$\therefore \quad p=\frac{P[\sin d\cos p+\cos d\sin p]}{R}$$

$$=\frac{P\left[\sin d\left(1-\frac{p^2}{2!}+\ldots\right)+\cos d\left(p-\frac{p^3}{3!}+\ldots\right)\right.}{R}$$

$$=\frac{P\sin d}{R}, \quad \text{as } Pp \text{ is very small:}$$

i.e., the common parallax is found by multiplying the greatest parallax by the sine of the zenith distance and dividing the product by the radius.*

The orbits, thus drawn, must be considered as *Drikkshepa Vrittas* or the azimuth circles for the nonagesimal.† The sine of the zenith distance of the nonagesimal or of the latitude of the zenith is the *Drikkshepa* of both the Sun and the Moon. Mark the nonagesimal points on the *Drikkshepa Vrittas* at the distance from the

* See note by Bapudev Sastri under verse 18, Golādhyāya, Chap. VIII.

† The point 90° distant along the ecliptic from the point on it just rising goes by the name of nonagesimal or *Trivalagna* or *Tribhona*.

zenith equal to the latitude of the points. From these two points (supposing them as the Sun and the Moon) find as before the minutes of parallax in latitude. These minutes are here *Nati-kalās*, *i.e.*, the minutes of the parallax in latitude of the Moon from the Sun. The difference north and south between the two orbits, *i.e.*, the measure of their mutual inclination, is the same in every part of the orbit as it is in the nonagesimal point, hence this difference called *Nati* is ascertained through the *Drikkshepa* or the sine of the zenith distance of the nonagesimal.

When the planet is depressed in the vertical circle, its north and south distance from its orbit caused by this depression is called *Nati* or the parallax in latitude.

Let Z be the zenith; N, the nonagesimal; ZNP, its vertical circle; Nsr, the ecliptic; P its pole; Zst, the vertical circle passing through the true place s and the depressed or apparent place t of the Sun; Ptr, a secondary circle to the ecliptic passing through the apparent place t of the Sun; then sr is the *Spasta Lambana* or parallax in longitude or tr the *Nati* or parallax in latitude which can be found in the following way:—

ZN = zenith distance of N, Zs = zenith distance of the Sun.

From $\triangle$ ZNs and tsr, sin Zs : sin ZN = sin st : sin rt.

$$\therefore \quad \sin rt = \frac{\sin st \times \sin ZN}{\sin Zs}.$$

Now, st is taken for sin st and rt for sin rt, on account of their being very small.

$$\therefore \quad rt = \frac{st \times \sin ZN}{\sin Zs}.$$

But we have already seen that $st = \dfrac{P \sin Zs}{R}$ ... (1)

$$\therefore \quad rt = \frac{P \sin ZN}{R} \qquad \dots (2)$$

i.e., the *Nati* is found by multiplying the sine of the latitude of the nonagesimal by the greatest parallax and dividing by the radius. It is clear that the north and south distance from the Sun depressed in

the vertical circle to the ecliptic, becomes equal to the common parallax at the nonagesimal and hence, the *Nati* is to be determined from the zenith distance of the nonagesimal. Hence by subtracting the *Nati* of the Sun from that of the Moon which are separately found in the above mentioned way, the parallax in latitude of the Moon from the Sun is found and this is equal to the difference between the mean parallaxes of the Sun and the Moon at the nonagesimal.*

The amount by which the Moon is depressed below the Sun deflected from the zenith (at the conjunction) where ever it be, is the east and west difference between the Sun and the Moon in a vertical circle For this reason, the difference is twofold, being partly east and west, and partly north and south. The ecliptic is here east and west and the circle secondary to it is north and south. (It follows from this, that the east and west difference lies in the ecliptic and the north and south difference in the secondary to it.) The difference east and west is the *Lambana* or parallax in longitude and the distance north and south is the *Nati* or parallax in latitude.

The parallax in minutes as observed in a vertical circle, forms the hypotenuse of a right angled triangle, of which the *Nati kalā* or the minutes of the parallax in latitude forms one of the sides adjoining the right angle, then the third side found by taking the square root of the difference of the squares of the two preceding sides will be *Sphuta Lambana Lipta* or the minutes of the parallax in longitude.

From the right angled triangle rst, rt=base, st=hypotenuse, sr= perpendicular $\therefore sr=\sqrt{(st)^2-(rt)^2}$

The amount in minutes of parallax in a vertical circle may be found by multiplying the sine of the Sun's zenith distance and the minutes of the extreme or horizontal parallax and dividing the product by the radius. Thus the *Nati* will be found from the *Drikkshepa* or the zenith distance of the nonagesimal.‡

Now when the sun's place is equal to the place of the nonagesimal, there is no parallax in longitude and that when the north latitude of the place is equal to the north declination of the nonagesimal point

* *Siddhānta Śiromaṇi*, Golādhyāya, Chap. VIII, verses 18, 19, 20.

† See notes by Bapudev Sastri under verses 18, 19, 20 of Golādhyāya.

At the time of the eclipse as the latitude of the Moon revolving in its orbit is very small, the Moon, therefore, is not far from the ecliptic, hence the parallax is determined from the corresponding place in the ecliptic, by neglecting the difference which is small.

‡ *Siddhānta Śiromaṇi*, Golādhyāya, Chap. VIII, verses 21, 22, 23, 24 and 25.

(*i.e.*, when the nonagesimal point is in the zenith of the place) there will be no parallax in latitude.*

When s coincides with N, sr, the longitude in parallax is zero and when Z coincides with N, tr, the latitude in parallax is zero.

In this connection the Sūrya Siddhānta has made use of some terms which are illustrated thus:

Let RMZPO represent the meridian, REO the projection of the horizon, QEC that of the equator, E the east point, Z the zenith, P the pole of the equator. Let rMNLΩ be the ecliptic, M its culminating or middle point (*madhya lagna*), L its *lagna* or rising point, N the nonagesimal (*triva lagna* or 90° apart from L in the ecliptic); K, the pole of the ecliptic, and HNZK the vertical circle passing through N, Hence r being assumed as vernal equinox, rM will represent the longitude of the *madhya lagna* or culminating point, rN that of the nonagesimal and rL of the rising point which is called the horoscope.

$$LH=ER, \therefore RH=EL$$

But EL represents the amplitude of the rising point, the sine of which or the *Udaya* is found thus:—Let QO and MT be the projections of the equator and the ecliptic and let V be the intersection of PE and MT; and let RSR be drawn perpendicular to EZ. Now form $\triangle$ EVL and QES, EL : EV :: EQ : ES or sine of the azimuth : sine of the declination :: radius : sine of altitude; and altitude = co-latitude of the place.

By the method of modern astronomy, cosPL=cosPOcosLO or sine (declination)=sine EL sin PZ, or, $\sin(\text{azimuth})=\dfrac{\sin(\text{dec.})}{(\sin \text{altitude})}$.

Also if rL is the equator and rs is the ecliptic, and S the Sun, then, $\dfrac{\sin rs}{\sin 90^\circ}=\dfrac{\sin SL}{\sin r}$, or, $\sin(\text{dec.})=\dfrac{\sin(\text{longitude})\times\sin 24^\circ}{\sin 90^\circ}$.

$=\dfrac{\sin(\text{longitude})\times\sin 24^\circ}{R}$, $\because \sin 90^\circ=$ Radius according to the Hindu astronomy.

$$\therefore \sin(\text{azimuth})=\frac{\sin(\text{dec.})\times\text{Radius}}{\sin(\text{altitude})}=\frac{\sin(\text{long.})\times\sin 24^\circ}{\sin(\text{altitude})},$$

* *Sūrya Siddhānta*, Chap V, verses 1 and 3.

Hence, $\quad Udaya=\dfrac{\sin \mathrm{L}\times\sin 24^\circ}{\cos l}$,

where L is the longitude and l, the latitude of the place.

Also MZ=QZ−QM or zenith distance of the culminating point is equal to QZ−its declination$=\mathrm{PO}-d=l-d$.

or meridian zenith distance of $\mathrm{M}=l\pm d$.

$\therefore$ sin (meridian zenith distance)$=\sin\,(l\pm d)$

or *Madhyajyā*$=\sin\,(l\pm d)$*

$$\text{Now } \sin\ \mathrm{MN}=\frac{\sin \mathrm{MZ}\ \sin \mathrm{MZN}}{\sin 90^\circ}=\frac{\sin \mathrm{MZ}\ \sin \mathrm{MZN}}{\mathrm{R}} \quad \ldots \quad (1)$$

[RH represents MZN]

$$=\frac{Madhyajy\bar{a}\times Udaya}{\mathrm{R}}=\frac{\sin\,(l\pm d)\times\sin \mathrm{L}\times\sin 24^\circ}{\mathrm{R}\cos l}.$$

The zenith distance NZ and the altitude NH of the nonagesimal point are found approximately from their sines, that of the zenith distance of N being called the *Drikkshepa*, that of the altitude *Driggati*.

Multiply the *Madhyajyā* by the *Udaya*, divide the product by the radius and square the quotient. Subtract the square from the square of the *Madhyajyā* ; the square root of the remainder is equal to the *Drikkshepa* or sine of the zenith distance of the nonagesimal, or the sine of the latitude of the zenith. The square root of the difference between the squares of the *Drikkshepa* and the radius is the sine of the altitude of the nonagesimal. The sine and cosine of the zenith distance of the culminating point are reckoned the rough *Drikkshepa* and *Driggati* respectively.†

From the right angled spherical triangle

$(\sin \mathrm{ZN})^2=(\sin \mathrm{ZM})^2-(\sin \mathrm{MN})^2$ or $\sin \mathrm{ZN}=\sqrt{\sin^2 \mathrm{ZM}-\sin^2 \mathrm{MN}}$

$$\text{or, } Drikkshepa=\sqrt{\sin^2\mathrm{ZM}-\left(\frac{\sin \mathrm{RM}\times\sin \mathrm{MZV}}{\mathrm{Q}}\right)^2}\ , \text{ from } (1)$$

Also, $\sin \mathrm{NH}=\sin(\mathrm{ZH}-\mathrm{ZN})=\sin \mathrm{ZH}\cos \mathrm{ZN}-\cos \mathrm{ZH}\sin \mathrm{ZN}$

* *Sūrya Siddhānta*, Chap. V, verses 4 & 5.

† *Ibid.*, verses 6 & 7.

But ZH=90°. cos ZH=0 and sin ZH=R

$\because$ sin NH=sin 90° cos ZN= $\sqrt{R^2 \cos^2 ZN} = \sqrt{R^2 - \sin^2 RN}$

$\therefore$ *Driggati* $= \sqrt{R^2 - (\textit{Drikkshepa})^2}$.

In the Sūrya Siddhānta we have the moon's horizontal parallax given in the form $\frac{5059}{224000} \times 3438'$ minutes of arc which reduced=53·681′. This can be found thus :—

Let OO′ be the Earth and MM′ the orbit of the Moon ; let E be the centre of the Earth, OM the horizontal line. ∠OME is the horizontal parallax.

$$\text{Now, } \frac{OE}{EM} = \frac{\angle OME}{\text{angular measure of EM}}$$

$$\therefore \text{ annual parallax} = \frac{OE}{EM} \times \text{angular measure of EM.}$$

With centre M and radius ME draw a circle. On the circumference of this circle take a point L such that EL=radius EM. Then the angle subtended at the centre M by this arc is the angular measure of EM and this is equivalent to unit radius=3438 minutes of arc (according to Sūrya Siddhānta).

$\therefore$ annual parallax

$$= \frac{\text{radius of the earth}}{\text{distance of the Moon from the centre of the Earth}}$$

Now, radius of the earth = 800 yojanas * and the circumference of the Moon's orbit is 324000 yojanas . $\therefore$ $EM = \frac{324000}{2\pi} = \frac{324000}{2\sqrt{10}}$

$$\therefore \text{ annual parallax} = \frac{800 \times 2\sqrt{10}}{324000} \times 3438' = \frac{5089}{324000} \times 3438' \text{ approx.,}$$

$$=53.681' \text{ approximately.}$$

This as a mean is smaller than what ought to be, as we know from modern astronomy the horizontal parallax is at the greatest=61.533′ and at the least=52.88′. The mean is 57.156′.

* *Sūrya Siddhānta*, Chap. I, verse 59.

But taking into consideration that the effects of refraction were not known in ancient India (and if known at all it was at a very recent date *), nor even in Europe till the time of Tycho Brahe and Kepler (the latter of whom gave the first treatise on refraction) it may be conceded that the Indian horizontal parallax of the Moon was a fair approximation.

As the horizontal parallax of the sun could not be obtained by direct observations, the Indian astronomers had recourse to the theory that all the planets moved in the respective orbits with the same actual linear velocity of about 12000 yojanas approximately in one day. By this hypothesis they accounted for the apparent slowness of some of them by their having to travel over orbits of greater diameter, and the circumferences were supposed to vary directly as the periodic times. Now the circumference of the moon's orbit is 324000 yojanas. Therefore the angular motion in one day is $\frac{12000}{324000} \times 360° = 13°20'$ approximately.

The horizontal parallax of moon is therefore approximately $\frac{1}{15}$ of its daily angular motion. $\left[53 \cdot 681' = \frac{1}{15} \text{of } 13°20' \text{ appox.}\right]$. The circumference of the sun's orbit is 4331500 yojanas. Therefore the daily angular motion of the sun $= \frac{12000}{4331500} \times 360° 59'$ approx. Hence the horizontal parallax of the sun $= \frac{1}{15} \times 59' = 3'56''$ approximatly. The relative parallax is therefore $48'46''$.

The equivalent in time of this is estimated to be 4 ghatikas, fifteenth part of a day.

The moon's parallax in longitude, on the occasion of a solar eclipse, involves a series of complex calculations, which are divided into steps. The true time of conjunction of the sun and the moon differs from the apparent time by the relative parallax at the sun and the moon expressed as time.

* There is one passage in the *Siddhāntatattva Viveka* by Kamalākarabhaṭṭa p. 86, verses 210-11, which might refer to refraction :

महामारीति माख्या वै ख्याता लोके प्रजायते
अथ बाष्पाम्बुग बिम्बमुदयेऽ कस्य पश्यति ॥
तेन ह्रस्वं भवेद् रक्तावर्णं भूपृष्ठगस्य तत् ।
पैत्ताशीनं वातश्रीनं खगतं यद्भुपादिकम् ॥

The first step is to compute a divisor called the chheda

$$=\frac{(\sin 30^\circ)^2}{Driggati}=\frac{R^2}{4\sin(\text{altitude of nonagesimal})}.$$

If the difference of the longitudes of the nonagesimal and of the sun be denoted by D, then the moon's parallax in longitude from the sun, expressed in ghatikas $=\frac{D}{Chheda}$. This will be a first approximation to the relative parallax in time.

The relative parallax in latitude of the moon from the sun is found thus: multiply the *Drikkshepa* (sine of zenith distance of nonagesimal) by the relative daily motion of the sun and moon and divide the product by fifteen times of the radius. Thus relative parallax in latitude $=\frac{48\frac{4}{5}}{3438}\times$ sine of the zenith's distance of the nonagesimal.

$$\therefore \text{ parallax in latitude}=\frac{\text{sine of the zenith distance of the nonagesimal}}{70}$$

$$=\frac{Drikkshepa}{70}.$$

Let MS be the ecliptic and Z the horizon. Let K be the pole of the ecliptic. Let S be the position of the sun when seen from the surface of the earth and T the position when seen from the centre of the earth. Draw the arc KT and KS Draw TR $\perp^r$ to KS. Produce arc KT to meet the ecliptic at V. Then TV is the parallax in latitude and VS is the parallax in longitude. Then $\frac{VS}{TR}=\frac{\sin KV}{\sin KT}=\frac{1}{\sin KT}$. $\therefore$ parallax in longitude $=VS=\frac{TR}{\sin KT}=\frac{TS\sin TSR}{\sin KT}$

$$=\frac{P\sin ZS}{R}\cdot\frac{\sin TSR}{\sin KT},$$

where P is the horizontal parallax and R, the radius,

$$=\frac{P\sin ZK\sin ZKS}{R\sin KT}$$

$$=\frac{P}{R}\times\frac{\sin(\text{altitude of the nonagesimal})}{\cos VT}\times\sin(ZKT+TKS).$$

But $\angle$ TKS is very small and cos VT=R approximately as VT is very small.

$$\therefore \text{ parallax in longitude} = \frac{P}{R} \cdot \frac{\sin (\text{alt. of nonagesimal})}{R} \sin ZKT$$

$$= P. \frac{\sin l. \sin r}{R^2},$$

where l=altitude and $r = \angle ZKT$=diff. of the longitudes of S and nonagesimal.

Let P=4 hours

$$\therefore \text{parallax in longitude} = \frac{4 \sin l \sin r}{R^2}$$

$$= \frac{\sin r}{\frac{(\frac{1}{2}R)^2}{\sin l}} = \frac{\sin r}{Chheda}, \text{ where, } \frac{(\frac{1}{2}R)^2}{\sin l} = Chheda.$$

Now as the horizontal parallax of the sun is $\frac{1}{15}$ of its daily motion and that of the moon is $\frac{1}{15}$ of its daily motion, therefore the difference

$= \frac{1}{15}$ (daily motion of the moon—daily motion of sun).

$\therefore$ parallax in latitude

$$= VT = TS \sin TSV$$

$$= \frac{P}{R} \sin ZS \cos ZSK = \frac{P}{R} \cos ZK$$

$$= \frac{P}{R} \sin (\text{zenith distance of the nonagesimal})$$

$$= \frac{Drikkshepa}{R}$$

$\times \frac{1}{15}$ (daily motion of the moon—daily motion of the sun)

$$= \frac{Drikkshepa}{15 \times 3438} (13^\circ \cdot 20' \cdot 46 \cdot 7''—59 \cdot 13616''), \text{ where } R=3435'$$

$$= \frac{48\frac{1}{2}}{3438} \times Drikkshepa \text{ approximately}$$

$$= \frac{Drikkshepa}{70} \text{ approximately.}$$

The annual parallax of a planet which is an angle subtended at the planet by the earth and the sun is also described in the Sūrya Siddhānta

in chapter II, verses 29 and 30 in connection with the finding of the mean places of the planets.

Next we shall consider in detail the procedure adopted in the old Sūrya Siddhānta as included in *Pañchasiddhāntikā*, for finding out the parallax.

We first find the *madhya-jyā*, *i.e.*, the sine of the zenith distance of the *madhya-lagna*, *i.e.*, that point of the ecliptic which at the time is on the meridian and then we find the sun's *Drikkshepa*.

Deduct the square of the *Drikkshepa* from the square of the Radius; multiply the square root of the remainder by the sine of the distance of the sun from the ecliptic point, and divide by the Radius, the result is the sine of the altitude of the sun. Let AC be the ecliptic; as, the moon's orbit, and P′ZTD, the projection of a great circle passing through the Pole of the Ecliptic, the zenith and the *tribhona* T, and cutting the moon's orbit in t. Let m be the true place of the sun and M the apparent place and P′m and P′M be secondaries; ZF be $\perp^r$ to P′m. sin Zm : sin (opposite angle) (which sine=sin $90°$=R)=sin ZF : sin opposite angle, and thus find the sine of $\angle$ ZmF which is equal to the cosine of $\angle$ TmZ. Now assuming mM to be the vertical parallax, and treating mMn as a plane right, angled triangle, $\angle$ mMn$=\angle$ TmZ; and $\angle$ mMn being the complement of $\angle$ Mmn, cos Mmn=sin mMn=cos TmZ.

$\therefore$ sin mnM(=R): sine of mM=sine $\angle$ mMn: sine of mn (*i.e.*, parallax in longitude).

Also R : sine of greatest vertical parallax=sine (given zenith distance): sine (desired vertical parallax); which (ZM being taken equal to Zm from which it differs very little) gives sine of desired vertical parallax

$$(\text{mM})=\frac{\sin\ (\textit{greatest parallax})}{\text{R}},\ \sin \text{Zm}.$$

The sine of the greatest vertical parallax being equal to the product of the earth's radius and R, divided by the planet's distance, and the abridged diameter of the earth being equal to 36, we have

$$\text{sine of greatest parallax}=\frac{18\times\text{R}}{\text{distance of planet}}.$$

$$\therefore\quad \sin\ \text{mM}=\frac{\text{sine greatest parallax}}{\text{R}}\times\sin\ \text{Zm}$$

$$=\frac{18\times\sin\ \text{Zm}'}{\text{distance of the planet}}.$$

$$\sin \angle mMn = \frac{\sin ZF \times R}{\sin Zm}.$$

Sine of parallax in longitude $= \sin mn = \dfrac{\sin mM \times \sin \angle mMn}{R}$

$$= \frac{18 \times \sin Zm}{\text{distance of planet}} \times \frac{\sin \angle mMn}{R} = \frac{18 \times \sin Zm}{\text{distance of planet} \times R}$$

$$\times \frac{\sin ZF \times R}{\sin Zm} = \frac{18 \sin ZF}{\text{distance of planet}}.$$

For the parallax in latitude, we get

R : sin (greatest parallax) = *drikkshepa* : sin (desired parallax in latitude)

$$\therefore \quad \sin \text{ (desired parallax)} \frac{18 \times R \times \textit{drikkshepa}}{R \times \text{true distance}} = \frac{18 \times \textit{drikkshepa}}{\text{true distance}}.*$$

In this connection Thibaut remarks in the introduction of his edition of *Pañchasiddhāntikā* that the rules laid down there for calculating solar and lunar eclipses agree with the modern rules of Sūrya Siddhānta as far as general methods are concerned, but at the same time there are some deviations in the details of the calculation of the parallax in solar eclipses. He admits that the main principles were there, only there have been changes in details.

Whitney, in course of a discussion on parallax in Hindu astronomy, says that the statement in the Sūrya Siddhānta that there is no parallax of the sun in longitude when the sun is in the meridian, is not true, unless the ecliptic is also bisected by the meridian.† But that is the implicit intention and it will be clear by perusing the original statement carefully that the meaning of the passage included that fact also.

There can be no doubt that from the time of the old Sūrya Siddhānta down to the time of *Siddhānta Śiromaṇi* various changes took place in the procedure for calculating the parallax. But it is certain that attempts to calculate the parallax were made from very early times and it is not true, as Mr. G. R. Kaye says,‡ that the attempt for the calculation of parallax in Hindu astronomy is but recent.

* *Pañchasiddhāntikā*, Chap IX, verses 23, 24, 25,

† *J. A. S. S.*, 1858, p. 286.

‡ *J. A. S. B.*, 1919, p. 177.

ON THE MEAN VALUE THEOREM OF THE DIFFERENTIAL CALCULUS

BY

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Introduction.

The *mean value theorem* of the Differential Calculus:

$$\frac{f(b)-f(a)}{b-a}=f'(a+\theta\overline{b-a}), \qquad 0<\theta<1$$

is usually enunciated to be valid subject to the condition that $f(x)$ possesses a differential coefficient at each point in the interval (a, b). This restriction as to the existence of a differential coefficient at each point within (a, b) was first removed by Prof. W. H. Young and Dr. G. C. Young, and the theorem has been proved to hold true in the following form: *

If there is no distinction of right and left † with regard to the derivates of $f(x)$, then there is a point in the completely open interval (a, b) at which $f(x)$ has a differential coefficient, and the value of that differential coefficient is precisely

$$\frac{f(b)-f(a)}{b-a}.$$

* W. H. Young and G. C. Young: On Derivates and the Theorem of the Mean, *Quarterly Journal of Mathematics*, Vol. 40, p. 10.

† The absence of the distinction of right and left at a point does not *necessarily* carry with it the continuity of the function at that point. For this reason, it is

In the present paper it has been shown that the above theorem holds even if the conditions imposed by Prof. W. H. Young and Dr. G. C Young are *further relaxed.* The new set of conditions is given in Theorem I. Theorem II of this paper may be termed Rolle's Theorem for Derivatives,* and ensures the existence of a determinate derivative at a point in the open interval (a,b), which is equal in value to the incrementary ratio of the end points of the interval, *i.e.*,

$$\frac{f(b)-f(a)}{b-a},$$

provided the derivates of $f(x)$ satisfy certain conditions. Theorems III and IV are the applications of Theorems I and II. Of these theorems, III is the generalisation of a theorem given by W. H. Young and G. C. Young.† Theorem V deals with the validity of the mean value theorem for discontinuous functions.

It is believed that all the theorems are new.

§1. THEOREM I. *If $f(x)$ be a continuous function defined in the closed interval (a, b), such that*

(1) there is no point within (a, b) at which one of the derivatives exists while the other does not, and

(2) at each point within (a, b) the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other; then, there exists a point in the completely open interval (a, b) at which the differential coefficient exists and its value is equal to

$$\frac{f(b)-f(a)}{b-a},$$

necessary to restrict this theorem to continuous functions. This will be clear from the following example :

Let $f(x)=\frac{3}{4}$ for $x=\frac{1}{2}$, $f(x)=\frac{1}{2}$ in the interval $(0, \frac{1}{2})$, and $f(x)=1$ in the interval $(\frac{1}{2}, 1)$.

There is no distinction of right and left at any point, and still the theorem of the mean does not hold for any interval containing $x=\frac{1}{2}$.

This seems to be due to the fact that the function is discontinuous.

* Following Prof. Young, throughout this paper, I use the word derivative to denote a one-sided differential coefficient.

† *loc. cit.*, p. 11, Cor. 2.

or in other words, there exists a value of θ for which

$$\frac{f(b)-f(a)}{b-a}=f'(a+\theta\overline{b-a}), \textit{ where } 0<\theta<1.$$

Proof:

Consider the function

$$F(x)=f(x)-f(a)-\frac{x-a}{b-a}\{f(b)-f(a)\}.$$

It is seen that $F(a)=F(b)=0$, so that $F(x)$ has at least one maximum or one minimum within (a, b).*

Suppose $f(x)$ has a maximum at the point ξ within (a, b).

Also by condition (2), the derivates at ξ satisfy either the inequality

$$\left.\begin{array}{l} D^+F(x)\geq D^-F(x) \\ D_+F(x)\leq D_-F(x) \end{array}\right\} \quad \ldots \quad \ldots \quad (A)$$

or

$$\left.\begin{array}{l} D^+F(x)\leq D^-F(x) \\ D_+F(x)\geq D_-F(x) \end{array}\right\} \quad \ldots \quad \ldots \quad (B)$$

Suppose that at the point ξ, the derivates satisfy the inequality (A). Now, as ξ is a maximum point,

$$D^+F(\xi)\leq 0, \textit{ and } D^-F(\xi)\geq 0 \quad \ldots \quad \ldots \quad (a)$$

But $D^+F(\xi)\geq D^-F(\xi)$ (by hypothesis A).

$$\therefore \quad D^+F(\xi)=D^-F(\xi)=0 \quad \ldots \quad \ldots \quad (b)$$

Again $D_-F(\xi)\geq 0$, for ξ is a maximum point.

But $D_-F(\xi)\leq D^-F(\xi)\leq 0$ by (b)

$$\therefore \quad D_-F(\xi)=0=D^-F(\xi)$$

We have now

$$F'_-(\xi)=0, \text{ and } D^+F(\xi)=0.$$

* For the trivial case $F(x)=$ Const., the theorem evidently holds.

Hence by condition (1) of the theorem, there exists a differential co-efficient at ξ

$$F'(\xi)=0.$$

Therefore

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

In a similar manner, if the derivates at ξ satisfy the inequality (B), it can be shown that

$$D^+F(\xi)=D_+F(\xi)=D_-F(\xi)=0,$$

so that by condition (1), $F'(\xi)$ exists and is equal to zero, and the theorem holds.

Similarly if ξ is a point of minimum it can be shown that the theorem holds.

We thus see that Theorem I. holds.

THEOREM II. *If $f(x)$ be a continuous function defined in the interval (a, b), such that the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other, then there is a point in the completely open interval (a, b) such that at least one of the two derivatives exists at this point and its value is*

$$\frac{f(b)-f(a)}{b-a}.$$

The theorem has already been proved by the reasoning adopted in Theorem I.

Ex. 1. *Let $\phi(x)$ equal $2x \sin\frac{1}{x}$ in the interval $(0, 1)$ and equal $x \sin \frac{1}{x}$ in the interval $(-1, 0)$.*

There is distinction of right and left as regards the derivates of $\phi(x)$ at the point $x=0$, so that Young's generalized condition for Rolle's Theorem is *not* satisfied for any interval containing the point $x=0$. Rolle's Theorem, however, holds for any interval containing $x=0$, for the conditions of Theorem I are satisfied. Here the differential co-efficient exists at each point of the interval except at $x=0$, where

$$D^+\phi(0)=2, \qquad D_+\phi(0)=-2,$$

$$D^-\phi(0)=1, \qquad D_-\phi(0)=-1;$$

so that the condition

$$\left.\begin{array}{l} D^+\phi(x) \geq D^-\phi(x) \\ D_+\phi(x) \leq D_-(x) \end{array}\right\}$$

is satisfied at each point of any interval containing $x=0$.

Ex. 2. Let $\psi(x)$ equal $2x \cos \log x^2$ in the interval $(0, 2)$, and equal $x \cos \log x^2$ in the interval $(-2, 0)$.

This example possesses the same properties as example 1 as regards the validity of Rolle's theorem.

Ex. 3. Let $$F(x) = \Sigma \frac{\psi(x-w_n)}{2^n},$$

where $\{w_n\}$ is an enumerable everywhere dense set of points in $(-1, 1)$ and $\psi(x)$ is the function defined in example 2.

At every point w_n there is distinction of right and left; Theorem I holds for any sub-interval of $(-1, 1)$.

§ 2. THEOREM III. *If $f(x)$ satisfies the conditions of Theorem I, then (1) there exists an everywhere dense set of points of potency c, where the differential coefficient of $f(x)$ exists, and (2) this differential coefficient passes through every value between its upper and lower limits.*

Proof:

By a well-known theorem, we know that the upper and lower limits of the derivates are the same as those of the incrementary ratio. Let these be U and L respectively, and let K be a number such that

$$L < K < U.$$

Then there is at least one interval (a_k, p_k) within (a, b) such that

$$\frac{f(p_k)-f(a_k)}{p_k-a_k} = K.$$

And as the conditions of Theorem I are satisfied by $f(x)$, there exists a point x_k in (a_k, p_k) such that there is a differential coefficient at x_k, and it has the value K, *i.e.*,

$$f'(x_k) = K.$$

But K may have any value between L and U, so that the points x_k, where the differential coefficients exist, are in (1, 1) correspondence with the continuum (L, U). Again, the conditions of theorem I are satisfied in every interval that can be taken within (a, b), so that the set of points $\{x_k\}$ is everywhere dense. Thus the set of points where the differential coefficient exists is everywhere dense and of the potency of the continuum.

Also, the differential coefficient can have every value K between L and U, so that it passes through all the values between its higher and lower limits.

THEOREM IV. *If $f(x)$ satisfies the conditions of theorem II, then (1) there is an everywhere dense set of points of potency c at which one of the derivatives exists, and (2) the derivative passes through all the values between its upper and lower limits.*

The proof of this theorem can be carried out as in Theorem III.

§ 3. THEOREM V. *If within (a, b), for which $f(x)$ is defined, there be a point ξ at which there is discontinuity of the second kind, at least on one side, say the right (left), and if the conditions of Theorem I are satisfied within a finite interval, however small, with the point of discontinuity ξ as left (right) end point, then there exists within (a, b) a point at which the differential coefficient exists and is equal to*

$$\frac{f(b)-f(a)}{b-a}.$$

It does not matter whether the conditions of Theorem I are satisfied at the other points of (a, b) or not. For instance, the function may be totally discontinuous, or may be non-differentiable at all the other points of (a, b).

Proof:

As the conditions of Theorem I are satisfied in an interval (ξ, α), lying within (a, b), it follows (by Theorem III of this paper) that the differential coefficient exists at an everywhere dense set in (ξ, α), and passes through all the values between the upper and lower limits of the incrementary ratio in (ξ, α). As there is discontinuity of the second kind at ξ, the upper and lower limits of the incrementary ratio are $+\infty$ and $-\infty$ respectively. Therefore, there exists at least one

point in (ξ, a), and consequently in (a, b_1), at which the differential coefficient exists and has any given value P. Hence there is at least one value of θ for which

$$\frac{f(b)-f(a)}{b-a}=f'(a+\theta\overline{b-a}),\ 0<\theta<1$$

and Theorem V holds.

Ex. 4. Let G be a non-dense set of points in (a, b). Let (α, β) be an interval complementary to the set G, and let $\phi(x, \alpha)=\sin\frac{1}{x-\alpha}$. Let $\alpha+\gamma$ be the greatest value of x, which does not exceed $\frac{1}{2}(\alpha+\beta)$, for which $\cos\frac{1}{x-\alpha}$ vanishes. Let $F(x)=0$ at every point of G, and in each interval (α, β) complementary to G, let

$$F(x)=\phi(x, \alpha), \quad \text{for } \alpha<x<\alpha+\gamma,$$

$$F(x)=\phi(\alpha+\gamma, \alpha), \quad \text{for } \alpha+\gamma\leq x\leq\beta-\gamma,$$

$$F(x)=-\phi(x, \beta), \quad \text{for } \beta-\gamma<x<\beta.$$

It is easy to see that any interval (p, q) taken within (a, b), contains within it a complementary interval of the set G, or is itself contained within such an interval. In the first case, the conditions of Theorem V are satisfied for the interval (p, q), and in the second case those of Theorem I. Hence the *mean value theorem* holds for any interval within (a, b), although $F(x)$ is discontinuous * at all the points of the set G, in (a, b).

My best thanks are due to Professor Ganesh Prasad for encouragement and interest.

* In a paper, "On a function which occurs in the law of the mean," published in the *Annals of Mathematics*, Vol. 7, 1906, pp. 177-192, Professor E. R. Hedrick writes: "That this formula" $f(x+h)-f(x)=hf'(x+\xi)$ "holds in a very general manner, in the precise form just given, even when $f(x)$ is discontinuous between x and $x+h$, is evident from such an example as $y=f(x)=\sin\frac{1}{x}$, which is essentially discontinuous."

Professor Hedrick seems to have been aware of the fact that the mean value theorem *may* hold, even if there is a discontinuity of the second kind within the interval. He, however, did *not* go beyond giving an example of a function which has such a discontinuity at a single point, but is continuous at all other points.

ROTATING ELLIPTIC CYLINDERS

BY

C. CHREE

In a recent paper in the *Bulletin* * Messrs. N. M. Basu and H. M. Sengupta give a solution of the elastic solid problem presented by a rotating isotropic elliptic cylinder. In their introductory remarks (*l.c.*, p. 141) they refer to Love's *Theory of Elasticity*, 2nd Ed., Art. 102 for the solution for a rotating circular cylinder, adding " but it appears that the problem of a cylinder whose cross section is any other than a circle has not yet been solved. In fact the equations are too complicated to admit of easy solution." The authors seemed to have overlooked a paper " Rotating elastic solid cylinders of elliptic section " which appeared in two parts in July and August, 1892, in the *Philosophical Magazine*. In this paper I obtained, in terms of ordinary Cartesian Co-ordinates solutions for rotating thin elliptic disks and long elliptic cylinders. These had the same degree of accuracy as the solutions I had previously obtained for rotating circular disks and cylinders, which formed the basis of the treatment in Love's *Theory of Elasticity* referred to by Messrs. Basu and Sengupta. The paper in the *Philosophical Magazine* investigated in considerable detail the way in which the strains and stresses vary with the ellipticity.

* Vol. XVIII, No. 3, p. 141.

ON THE SUMMABILITY (C1) OF THE FOURIER SERIES OF A FUNCTION AT A POINT WHERE THE FUNCTION HAS AN INFINITE DISCONTINUITY OF THE SECOND KIND.

BY

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The object of the present paper is to apply Paul du Bois-Reymond's Infinitärcalcül to obtain as complete an answer as possible to the following question: When is the Fourier series of a function $f(x)$ summable (C1) at a point x_0, where $f(x)$ has an infinite discontinuity of the second kind such that

$$\phi(t) \equiv f(x_0+2t)+f(x_0-2t)=\chi(t) \cos \psi(t),$$

$\chi(t)$ and $\psi(t)$ being both monotone functions which tend to infinity as t tends to 0? In answering this question, I show, among other things, that the series may be summable at x_0 even if the indefinite integral of $\phi(t)$ is *not* differentiable at $t=0$

In a recent paper,* I mentioned the aforesaid question which does not seem to have been attacked *directly* by any previous writer.†

It may be noted, as is well-known, that Cesàro's mean $S_n(x_0)$ behaves for large values of n as

$$\frac{1}{n\pi}\int_0^a \left(\frac{\sin nt}{t}\right)^2 \phi(t)\, dt,$$

* This *Bulletin*, Vol. 18, footnote on p. 184.

† The most recent investigation in this connection is probably Pollard's (See his paper, "Direct proof of a new criterion for the summability of a Fourier series," *Journal L. M. S.*, Vol. I, 1926, pp. 233-235). The criterion considered by Pollard *fails in the case treated by me.*

where $a>0$ is independent of n and can be chosen to be as small as we please. Thus, integrating the above by parts, $S_n(x_0)$ behaves as

$$\frac{1}{n\pi}\left(\frac{\sin na}{a}\right)^2\Phi_1(a)+\frac{2}{n\pi}\int_0^a\frac{\Phi_1(t)}{t}\left(\frac{\sin nt}{t}\right)^2 dt$$

$$-\frac{1}{\pi}\int_0^a\frac{\Phi_1(t)}{t}\;\frac{\sin 2nt}{t}\,dt,$$

$\Phi_1(t)$ standing for

$$\int_0^t \phi(t_1)\,dt_1.$$

Hence, neglecting terms, which vanish with $\frac{1}{n}$,

$$S_n(x_0)=\frac{2}{n\pi}\int_0^a\frac{\Phi_1(t)}{t}\left(\frac{\sin nt}{t}\right)^2 dt-\frac{1}{\pi}\int_0^a\frac{\Phi_1(t)}{t}\;\frac{\sin 2nt}{t}\,dt. \quad \text{(A)}$$

1. The Fourier series of $f(x)$ is summable (C1) at x_0, when

$$\psi \succ \log\frac{1}{t^2} \text{ and } \frac{\chi}{(t\psi')} \preceq 1\,;$$

and it is not summable (C1) at x_0, when

$$\psi \preceq \log\frac{1}{t^2}\,.$$

Proof:

(*a*) Let $\quad \frac{\chi}{(t\psi')} \preceq 1$. Then, since $\chi \succ 1$,

$$\psi' \succ \frac{1}{t}\,,$$

i.e., $\quad \psi \succ \log\frac{1}{t^2}$

Also $\Phi_1(t)$ behaves* as $\frac{\chi}{\psi'}$ $\sin\psi$.

Therefore, when

$$\frac{\chi}{t\psi'} \prec 1,$$

$$\frac{\Phi_1'(t)}{t} \prec 1,$$

and, consequently, both the parts of (A) tend to zero with $\frac{1}{n}$; thus in this case

$$\operatorname*{Lim}_{n=\infty} \ S_n(x_0)=0.$$

Again, when†

$$\frac{\chi}{t\psi'} \asymp 1,$$

$\frac{\Phi_1(t)}{t}$ behaves as $\sin\psi(t)$ where $\psi \succ \log\frac{1}{t^2}$; hence each part of (A) tends to O with $\frac{1}{n}$. Therefore in this case too

$$\operatorname*{Lim}_{n=\infty} \ S_n(x_0)=0.$$

(*b*) Let $\psi \preceq \log\frac{1}{t^2}$.

Then it is known‡ that

$$\frac{1}{n\pi}\int_0^a \chi\cos\psi\left(\frac{\sin nt}{t}\right)^2 dt$$

* It is easy to prove that, when $\psi \succ \log\frac{1}{t^2}$, Φ_1 behaves as $\frac{\chi}{\psi'}$ $\sin\psi$ whether $\frac{\chi}{\psi'} \preceq t$ or $\succ t$. *e.g.*, take the case $\frac{\chi}{\psi'} \prec t$; here, integrating by parts,

$$\int_0^t \phi dt = \frac{\chi}{\psi'}\sin\psi - \int_0^t \frac{d}{dt}\left(\frac{\chi}{\psi'}\right)\sin\psi\, dt. \text{ As } \frac{\chi}{\psi'} \prec t, \frac{d}{dt}\frac{\chi}{\psi'} \prec 1$$

Therefore Φ_1 behaves as $\frac{\chi}{\psi'}\sin\psi + \frac{\cos\psi}{\psi'}\frac{d}{dt}\left(\frac{\chi}{\psi'}\right) + $, *i.e.* as $\frac{\chi}{\psi'}\sin\psi$.

† Note that in this case $\Phi'_1(0)$ is non-existent.

‡ See Hardy's paper, "Oscillating Dirichlet's integrals" (*Quarterly Journal of Mathematics*, Vol. 44, 1913), p. 35.

behaves as

$$\chi\left(\frac{1}{n}\right)\cos\psi\left(\frac{1}{n}\right).$$

Therefore

$$\operatorname*{Lim}_{n=\infty} S_n(x_0) \text{ is non-existent.}$$

Ex. 1. For $\chi=\frac{1}{t^k}$, $\psi=\frac{1}{t^l}$, where $l \geq k > 0$, the series is summable.

Ex. 2. For $\chi=1\Big/\left(t\log\frac{1}{t^2}\right)$, $\psi=\log\frac{1}{t^2}$, the series is not summable.

2. If $\psi \succ \log\frac{1}{t^2}$ and $\frac{\chi}{t\psi'} \succ 1$ but $\frac{\chi}{(t\psi')^2} \precsim 1$, then the Fourier series of $f(x)$ is summable (C1) at x_0 when

$$\frac{\chi}{(t\psi')} \prec t\sqrt{\psi''};$$

the series is not summable when

$$\frac{\chi}{(t\psi')} \succ t\sqrt{\psi''}.$$

Proof :

Here $\Phi_1(t)$ behaves as $\frac{\chi}{\psi'}\sin\psi$. Therefore, by the theorem of the preceding article, the first part of (A) tends to 0 with $\frac{1}{n}$, for

$$\frac{\chi}{t\psi'}\Big/ t\psi' \precsim 1,$$

i.e., $$\frac{\chi}{(t\psi')^2} \precsim 1.$$

Thus the behaviour of $S_n(x_0)$ depends in this case on the second integral of (A) which, by Du Bois-Reymond's result, converges or diverges according as

$$\frac{\chi}{t\psi'} \prec t\sqrt{\psi''}$$

or

$$\frac{\chi}{t\psi'} \succ t\sqrt{\psi''}\ .$$

Ex. 3. For $\chi = \frac{1}{t^2 L^5}$, $\psi = \frac{1}{t^{1/3}}$, the series is summable

Ex. 4. For $\chi = \frac{1}{t^{8/5}}$, $\psi = \frac{1}{t^{1/3}}$, the series is not summable.

3. If $\psi \succ \log \frac{1}{t^2}$ *and* $\frac{\chi}{(t\psi')^2} \succ 1$ but $\frac{\chi}{(t\psi')^3} \preceq 1$, then the Fourier series of $f(x)$ is not summable (C1) at x_0 when

$$\frac{\chi}{(t\psi')^2} \prec t\sqrt{\psi''}.$$

Proof :

Integrating by parts the first integral in (A) and neglecting terms which vanish with $\frac{1}{n}$ we have

$$S_n(x_0) = \frac{4}{n\pi}\int_0^a \frac{\Phi_2(t)}{t}\left(\frac{\sin nt}{t}\right)^2 dt$$

$$-\frac{2}{\pi}\int_0^a \frac{\Phi_2(t)}{t}\ \frac{\sin 2nt}{t}\ dt - \frac{1}{\pi}\int_0^a \frac{\Phi_1(t)}{t}\ \frac{\sin 2nt}{t} dt, \qquad \text{(B)}$$

where $\Phi_2(t) = \displaystyle\int_0^t \frac{\Phi_1(t_1)}{t_1}\ dt_1.$

Now $\Phi_2(t)$ behaves as $\left(-\frac{\chi}{t\psi'}\Big/\psi'\right)\cos\psi$.

Therefore $\frac{\Phi_2(t)}{t}$ behaves as $\frac{\chi}{(t\psi')^2}\cos\psi$.

Hence, applying the theorem of Art. 1 to the first part of (B), that part tends to o with $\frac{1}{n}$, as

$$\frac{\chi}{(t\psi')^2}\Big/ t\psi' \precsim 1.$$

Thus the behaviour of S_n depends on

$$\frac{1}{\pi}\int_o^a \frac{2\Phi_2+\Phi_1}{t}\;\frac{\sin 2nt}{t}\;dt. \qquad 1)$$

But $\qquad \psi \succ \log\frac{1}{t^2}$,

and, consequently, $\frac{1}{\psi'} \prec t$

and therefore $\frac{\psi''}{(\psi')^2} \prec 1$, *i.e.*, $\sqrt{\psi''} \prec \psi'$.

Hence $\frac{\chi}{t\psi'} \succ t\sqrt{\psi''}$,

since by hypothesis, $\frac{\chi}{t\psi'} \succ t\psi'$.

Also, by hypothesis $\frac{\chi}{(t\psi')^2} \prec t\sqrt{\psi''}$,

Hence, by Du Bois-Reymond's result, the part of (1) due to $\frac{2\Phi_2}{t}$ converges with increasing n and that due to $\frac{\Phi_1}{t}$ does not converge.

$$\text{Therefore} \quad \underset{n=\infty}{\text{Lim}}\ S_n(x_0) \text{ is non-existent.}$$

Ex. 5. For $\chi = t^{-\frac{9}{16}}$, $\psi = t^{-\frac{1}{4}}$, the series is not summable.

4.* If $\psi \succ \log \frac{1}{t^2}$ and $\frac{\chi}{(t\psi')^m} \succ 1$ but $\frac{\chi}{(t\psi')^{m+1}} \preceq 1$, m being greater than 1 ; then the Fourier series of $f(x)$ is summable (C1) at x_0 according as

$$\underset{n=\infty}{\text{Lim}} \int_0^a \frac{\sin 2nt}{t} \left(\frac{\Phi_1 + 2\Phi_2 + 2^2\Phi_3 + \ldots + 2^{m-1}\Phi_m}{t} \right) dt \quad (1)$$

is existent or not, $\Phi_r(t)$ standing for

$$\int_0^t \frac{\Phi_{r-1}(t_1)}{t_1} dt.$$

Proof :

Applying integration by parts $(m-1)$ times to the first integral in (A) and neglecting terms which vanish with $\frac{1}{n}$, we have

$$S_n(x_0) = \frac{2^m}{n\pi} \int_0^a \frac{\Phi_m(t)}{t} \left(\frac{\sin nt}{t} \right)^2 dt$$

$$- \frac{1}{\pi} \int_0^a \frac{\sin 2nt}{t} \left(\frac{\Phi_1 + 2\Phi_2 + \ldots + 2^{m-1}\Phi_m}{t} \right) dt.$$

* By means of this theorem, the uncertainty in this case about the answer to the question of this paper is shown to be due to our insufficient knowledge of the behaviour of oscillating Dirichlet's integrals ; *e.g.*, it is *doubtful* if (1) is *always* non-existent when each part of it due to Φ_1, Φ_2, etc. is non-existent.

Hence, reasoning as in the preceding article, the result follows; since $\frac{\chi}{(t\psi')^r} \succ t\sqrt{\psi''}$ for $r=1, 2, \ldots m-1$, and consequently the parts due to $\Phi_1, \Phi_2, \ldots \Phi_{m-1}$ are all non-existent.

Ex. 6. For $\chi = t^{-\frac{13}{16}}$, $\psi = t^{-\frac{1}{4}}$, the series is *probably* not summable.

Bull. Cal. Math. Soc. Vol. XIX, No. 2, (1928).

2

THE STEADY ROTATIONAL MOTION OF A LIQUID WITHIN FIXED BOUNDARIES.

BY

B. M. SEN.

(*Calcutta*)

It is well-known that no irrotational motion of a liquid contained within a fixed boundary, or outside a fixed boundary and at rest at infinity is possible. But if the flux across the boundary be prescribed, the motion can be uniquely determined. In the case of rotational motion the form of the boundary is connected with the possible distribution of vorticity. It is proposed in the present paper to investigate this relation.

The related problem of motion of cylinders and spheres in a liquid of constant vorticity has been investigated by Proudman.*

1. Clebsch has expressed the velocity components of a liquid moving irrotationally in the form

$$u=-\frac{\partial \phi}{\partial x}+\lambda\frac{\partial \mu}{\partial x},$$

$$v=-\frac{\partial \phi}{\partial y}+\lambda\frac{\partial \mu}{\partial y},$$

$$w=-\frac{\partial \phi}{\partial z}+\lambda\frac{\partial \mu}{\partial z} \qquad \dots \quad (1)$$

The rotation components are

$$\xi=\frac{\partial(\lambda, \mu)}{\partial(y, z)}, \quad \eta=\frac{\partial(\lambda, \mu)}{\partial(z, x)}, \quad \zeta=\frac{\partial(\lambda, \mu)}{\partial(x, y)}. \qquad \dots \quad (2)$$

* *Proc. Roy. Soc.*, xcii, 408, and S. F. Grace, *Proc. Roy. Soc.*, cii, 89.

It has also been shown that the vortex-lines lie on the surfaces

$$\lambda=\text{const.}, \qquad \mu=\text{const.} \qquad \dots \quad (3)$$

The dynamical equations can then be regarded as giving the pressure in terms of the functions ϕ, λ, μ.

It is obvious from the forms of the expressions for the velocities that λ is indeterminate to the extent of a factor of the form $f(\mu)$. We can therefore, choose it so that

$$\lambda=\text{const., and } \mu=\text{const.}$$

represent two orthogonal families of surfaces.

Then the equation of continuity gives for an incompressible liquid

$$-\nabla^2\phi+\lambda\nabla^2\mu=0. \qquad \dots \quad (4)$$

If the fixed boundary be given by

$$\psi = 0,$$

the condition of zero normal velocity gives

$$\frac{\partial\psi}{\partial x}\left\{-\frac{\partial\phi}{\partial x}+\lambda\frac{\partial\mu}{\partial x}\right\}+\frac{\partial\psi}{\partial y}\left\{-\frac{\partial\phi}{\partial y}+\lambda\frac{\partial\mu}{\partial y}\right\}$$

$$+\frac{\partial\psi}{\partial z}\left\{-\frac{\partial\phi}{\partial z}+\lambda\frac{\partial\mu}{\partial z}\right\}=0. \qquad (5)$$

One way of satisfying this relation would be to take ψ orthogonal to the families

$$\phi=\text{const.}, \qquad \mu=\text{const.}$$

Also we have taken λ and μ surfaces mutually orthogonal. The following then are the possibilities :

(A) ψ may be orthogonal to the families λ, μ.

ϕ orthogonal to ψ, so that $\phi=\lambda+\kappa\mu$.

(B) ψ may be one of the family λ.

Then λ, μ, ϕ form a triply orthogonal family of surfaces.

2. For the simplest case of steady two-dimensional motion, we may take

$$\phi=\text{const. in (B)}$$

$$\lambda=f(\varpi),\quad \mu=\theta,\quad \text{and}\quad \phi=\text{const.}$$

Then

$$\zeta=\frac{\partial(\lambda,\mu)}{\partial(x,y)}=\frac{f'}{\varpi} \qquad \dots \quad (6)$$

$$u=-\frac{fy}{\varpi^2},\quad v=\frac{fx}{\varpi^2}. \qquad \dots \quad (7)$$

The dynamical equations become, for steady motion,

$$\left.\begin{aligned} -v\zeta&=-\frac{ff'x}{\varpi^3}=-\frac{\partial\chi}{\partial x}\\ u\zeta&=-\frac{ff'y}{\varpi^3}=-\frac{\partial\chi}{\partial y}\end{aligned}\right\} \qquad \dots \quad (8)$$

where $\chi=\frac{p}{\rho}+\frac{1}{2}q^2+\Omega.$

The condition of integrability is satisfied and

$$\chi=\int\frac{ff'}{\varpi^2}d\varpi \qquad \dots \quad (9)$$

We have, therefore,

$$\frac{p}{\rho}=\int\frac{ff'}{\varpi^2}d\varpi-\frac{1}{2}\frac{f^2}{\varpi^2}-\Omega+\text{const.} \qquad \dots \quad (10)$$

If Ω be symmetrical round the z-axis, p is symmetrical round the axis and depends on ϖ. The resultant thrust on the cylinder is zero.

3. Reverting to the general case of steady two-dimensional motion, the dynamical equations may be written

$$\left.\begin{aligned} -v\zeta &= -\frac{\partial \chi}{\partial x} \\ u\zeta &= -\frac{\partial \chi}{\partial y} \end{aligned}\right\} \quad \dots \quad (11)$$

where $\chi = \frac{p}{\rho} + \frac{1}{2} q_1^2 + \Omega.$

Also $$\zeta = \frac{\partial(\lambda, \mu)}{\partial(x, y)} = \lambda_1 \mu_2 - \lambda_2 \mu_1 \quad \dots \quad (12)$$

the suffixes 1, 2 denoting differentiation with respect to x, y respectively.

The condition of orthogonality may be written

$$\lambda_1 \mu_1 + \lambda_2 \mu_2 = 0 \quad \dots \quad (13)$$

which gives

$$\left.\begin{aligned} \lambda_{11}\mu_1 + \lambda_1\mu_{11} + \lambda_{12}\mu_2 + \lambda_2\mu_{12} &= 0 \\ \lambda_{12}\mu_1 + \lambda_1\mu_{12} + \lambda_{22}\mu_2 + \lambda_2\mu_{22} &= 0 \end{aligned}\right\} \quad \dots \quad (14)$$

The condition of integrability gives

$$\frac{\partial}{\partial y}(v\,\zeta) + \frac{\partial}{\partial x}(u\,\zeta) = 0$$

or,

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = 0 \quad \dots \quad (15)$$

leading to

$$\mu_1\{\lambda_{11}\mu_2 + \lambda_1\mu_{12} - \lambda_{12}\mu_1 - \lambda_2\mu_{11}\}$$
$$+\mu_2\{\lambda_{12}\mu_2 + \lambda_1\mu_{22} - \lambda_{22}\mu_1 - \lambda_2\mu_{12}\} = 0 \quad \dots \quad (16)$$

Using equations (14), this can be reduced to the form

$$(\mu_{22} - \mu_{11})(\mu_2^2 - \mu_1^2) + 4\mu_{12}\mu_1\mu_2 = 0 \quad \dots \quad (17)$$

In the usual notation of partial differential equations

$$(t-r)(q^2-p^2)+4pqs=0 \qquad \dots \quad (18)$$

4. This equation belongs to Monge's type of differential equations with two intermediary integrals. In accordance with the usual method, they are given by

$$\left.\begin{aligned}(p^2-q^2)\,(dx^2-dy^2)+4\,pq\,dx\,dy=0\\ dp\,dy-dq\,dx=0\end{aligned}\right\} \qquad \dots \quad (19)$$

which give

$$\{(p+q)dx-(p-q)dy\}\{(p-q)dx+(p+q)dy\}. \qquad \dots \quad (20)$$

The first intermediary integral is given by

$$\left.\begin{aligned}(p+q)dx-(p-q)dy=0\\ dp\,dy-dq\,dx=0\end{aligned}\right\} \qquad \dots \quad (21)$$

One integral of these equations is

$$(p^2+q^2)^{\frac{1}{2}}=\mathrm{A}e^{\tan^{-1}q/p} \qquad \dots \quad (22)$$

Again $\quad x+ny=$const. is another integral, provided $n=\pm i$.

By Monge's method, one intermediary integral is

$$e^{\tan^{-1}p/q}=f(x\pm iy)\,(p^2+q^2)^{\frac{1}{2}}. \qquad \dots \quad (23)$$

Similarly the second intermediary integral is

$$e^{\tan^{-1}p/q}=\mathrm{F}(x\pm iy)\,(p^2+q^2)^{\frac{1}{2}}. \qquad \dots \quad (24)$$

We deduce from (23), (24)

$$e^{\pi/2}=f(x\pm iy)\,\mathrm{F}(x\pm iy)\,(p^2+q^2). \qquad \dots \quad (25)$$

For real values of (p, q), the functions f and F must be conjugate functions. We take these as

$$f=f(x+iy),\quad \mathrm{F}=f(x-iy)$$

The only real solution is

$$p=\text{const. and } q=\text{const.}$$

which are equivalent to the rather trivial solution

$$\mu=x,\quad \lambda=y.$$

It thus appears that Monge's method* which does not, of course, give the most general solution is of no use. It has not been found possible to obtain a more general solution.

5. Some particular cases of two-dimensional motion can however be obtained more easily by assuming

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}.$$

The fixed boundary must be one of the stream lines $\psi = \text{const}$.

The equations of motion are integrable if

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = 0$$

or,

$$\frac{\partial(\psi, w)}{\partial(x, y)} = 0 \qquad \dots \quad (26)$$

We have, therefore, the well-known equation*

$$\nabla_1^2 \psi + f(\psi) = 0 \qquad \dots \quad (27)$$

Solutions of this equation can be obtained for only a few particular forms of the arbitrary function f. Putting $f(\psi) = -1$, we get

$$\psi = ax^2 + 2cxy + by^2 \qquad \dots \quad (28)$$

where $2(a+b) = 1$.

If we put $f(\psi) = -k^2\psi$, we get Stoke's solution for a circular boundary.

The value

$$f(\psi) = k^2\psi$$

leads to the solution

$$\psi = \sin\frac{x\pi}{a}\sin\frac{\lambda\pi}{b}, \quad \text{where } k^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \qquad \dots \quad (29)$$

corresponding to the rectangular boundary $x = \pm a$, $\lambda = \pm b$.

* Stokes, *Papers*, i. 15.

6. The equation (27) may be transformed with the substitution (with a slight change of notation)

$$x+iy=\xi,\ x-iy=\eta$$

into

$$\frac{\partial^2\psi}{\partial\xi\partial\eta}=f(\psi) \qquad \dots \quad (30)$$

or

$$s=f(\psi).$$

This is equivalent to

$$\frac{\partial s}{\partial\xi}\frac{\partial\psi}{\partial\eta}=\frac{\partial s}{\partial\eta}\frac{\partial\psi}{\partial\xi}$$

or to

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{\partial\psi}{\partial\eta}\right)^2=\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial\psi}{\partial\xi}\right)^2. \qquad \dots \quad (31)$$

It may be pointed out that this equation which gives the motion is of the third order whereas the equation (18) previously found is of the second.

It may thus be possible to discover new solutions of problems.

7. The steady three-dimensional motion with uniform vorticity is easily reduced to the case of two-dimensional motion with uniform motion in the third direction superposed. The solution therefore depends on equation (17).

8. If the vorticity be constant in direction but not uniform, we may take,

$$\xi=\eta=0,$$

and

$$u=-\frac{\partial\phi}{\partial x}+\frac{\partial H}{\partial y}$$

$$v=-\frac{\partial\phi}{\partial y}-\frac{\partial H}{\partial x}$$

and

$$w=-\frac{\partial\phi}{\partial z}, \qquad (32)$$

where H is a function of x, y only.

Then $\zeta=-\nabla_1{}^2 H$, and $\nabla^2\phi=0$.

The equations of motion become

$$\left.\begin{aligned} -v\zeta &= -\frac{\partial \chi}{\partial x} \\ u\zeta &= -\frac{\partial \chi}{\partial y}. \end{aligned}\right\}$$

The condition of integrability is

$$\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=0$$

or

$$\zeta\frac{\partial w}{\partial z}=u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}$$

or

$$\frac{\partial w}{\partial z}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\log \zeta.$$

This is an equation connecting ϕ and H. If this can be boundary is given by

$$u\frac{\partial \chi}{\partial x}+v\frac{\partial \chi}{\partial y}+w\frac{\partial \chi}{\partial z}=0$$

which simplifies on substitution

$$\chi=\log \zeta+Z$$

$$\frac{\partial w}{\partial z}=-w\frac{\partial Z}{\partial z},$$

when Z is a function of z only.

We have, therefore,

$$Z+\log w=f(x, y)$$

If a solution of equation (35) be known, the correspondi can be found.

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ON THE GRAVITATIONAL FIELD OF AN IDEAL FLUID.

BY

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I. *Introduction.*

§1. The gravitational equations of Einstein for the fields occupied by matter have been solved only in a few cases relating to fluids at rest. For such a medium, it is assumed that all the oblique components of the material-energy-tensor T_μ^ν vanish, while three of the remaining components, which are identified with the pressures of the fluid, are equal. There seems to exist a certain amount of difference of opinion as regards the interpretation of the fourth surviving component (T_4^4), and the assumption of the equation of state. Schwarzschild* solved the problem for a medium which he defined by identifying T_4^4 with the density and whose incompressibility was expressed by the constancy of T_4^4. The density has been identified with T_4^4 also by Combridge.† Schrödinger‡ adopted the assumption $\rho-p=T_4^4=3T_1^1=3T_2^2=3T_3^3=-3p$, where p and ρ represent the pressure and density of the fluid respectively The gravitational equations were solved by Bauer§ for fluids whose equations of state were defined by him by the assumptions $\rho-p=$const. and $\rho+2p=$const., where $\rho-p=T_4^4$ and $-p=T_1^1=T_2^2=T_3^3$. His results include the solution of Schwarzschild, which, however, according to Eddington,|| does not represent the solution for an incompressible

* *Berliner Sitzungsberichte*, 1916, p. 424.

† The field of a spherical shell on the relativity theory, *Phil. Mag.*, Jan., 1926.

‡ *Phys. Zeitschrift*, 19 (1918).

§ Kugelsymmetrische Lösungssysteme der Einsteinschen Feldgleichungen der Gravitation usw., *Wiener Sitz.*, IIa, Bd. 127 (1918), p. 2141.

|| *Mathematical Theory of Relativity*, Arts. 54, 72.

liquid sphere, because the condition for incompressibility should be $T=\sum_\nu T_\nu^\nu=\rho-4p=$const. and not $T_4^4=\rho=$const. The equations considered by Schwarzschild, Schrödinger and Bauer are the original gravitational equations in which the cosmological constant is zero.

The object of the present paper is to investigate the gravitational field of a mass of fluid of ideal compressibility, *viz.*, one in which the equation of state is expressed by the equality of the pressure and density at every point, the field-equations being assumed to have either the original form or the form containing the cosmological constant. The solution in the former case agrees in form with that of Bauer for the case $\rho-p=0$, obtained by a different method, but he has not considered the case $\rho=p\neq$const., which is the fundamental assumption of the problem under our consideration.

The vanishing of the component T_4^4, which is sometimes called the energy-density, may perhaps be objected to on the ground that it is inconsistent with matter, but the objection does not seem to be valid, because the presence of matter is sufficiently indicated by the existence of the material-energy-invariant T, which is here a non-vanishing scalar depending on the physical property of the medium.

II. *The Equations.*

§2. The gravitational equations are

$$K_\nu^\sigma-\tfrac{1}{2}g_\nu^\sigma(K-2\beta)=-8\pi T_\nu^\sigma \quad \dots \quad (1)$$

where $K_{\mu\nu}$ is the contracted Riemann-Christoffel tensor, T_ν^σ is the material energy-tensor, and β is the cosmological constant. We assume the matter to be distributed symmetrically about the origin, so that the metric of space-time is

$$ds^2=-e^\lambda dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^\nu dt^2, \quad (2)$$

λ, ν being functions of r only.

The equations (1) then become

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\nu}{dr}-\frac{e^\lambda-1}{r^2}\right)=-8\pi\left(T_1^1+\frac{\beta}{8\pi}\right), \text{ or } -8\pi T_1^1 \quad \dots \quad (3)$$

$$e^{-\lambda}\left\{\frac{1}{2}\frac{d^2\nu}{dr^2}-\frac{1}{4}\frac{d\lambda}{dr}\frac{d\nu}{dr}+\frac{1}{4}\left(\frac{d\nu}{dr}\right)^2+\frac{1}{2r}\left(\frac{d\nu}{dr}-\frac{d\lambda}{dr}\right)\right\}$$

$$=-8\pi\left(T_2^2+\frac{\beta}{8\pi}\right), \text{ or } -8\pi T_2^2 \quad \dots \quad (4)$$

$$T_3^3=T_2^2 \quad \dots \quad (5)$$

$$e^{-\lambda}\left(-\frac{1}{r}\frac{d\lambda}{dr}-\frac{e^{\lambda}-1}{r^2}\right)=-8\pi\left(T_4^4+\frac{\beta}{8\pi}\right), \text{ or } -8\pi T_4^4 \dots \quad (6)$$

the second alternative corresponding to the form in which the cosmological constant vanishes.

§ 3. Let ρ and p represent the density and pressure of the fluid respectively. The material energy tensor is then given by

$$T_\nu^\sigma=\delta-\delta_\nu^\sigma p+g_{p\sigma}\frac{dx_q}{ds}\frac{dx^\sigma}{ds}\rho, \quad \dots \quad (7)$$

so that the components in a galilean system are

$$T_\nu^\sigma = \begin{matrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & \rho-p \end{matrix}$$

We now *define* a perfect fluid of ideal compressibility by the equation

$$\rho-p=0 \quad \dots \quad (8)$$

at every point. In this case, therefore, we have

$$T_1^1=T_2^2=T_3^3=-p,\ T_4^4=0$$

$$T=T_\nu^\nu=-3p.$$

III. *The field of an ideal fluid according to the original law of gravitation.*

§ 4. We take the second form of the equations (3)...(6), and we obtain from (6),

$$\frac{1}{r}\frac{d\lambda}{dr}+\frac{e^{\lambda}-1}{r^2}=0 \qquad \dots \quad (9)$$

From (3) and (4), since $T_1^1=T_2^2$, we have

$$\frac{1}{2}\frac{d^2\nu}{dr^2}-\frac{1}{4}\frac{d\lambda}{dr}\frac{d\nu}{dr}+\frac{1}{4}\left(\frac{d\nu}{dr}\right)^2+\frac{1}{2r}\left(\frac{d\nu}{dr}-\frac{d\lambda}{dr}\right)$$

$$-\frac{1}{r}\frac{d\nu}{dr}+\frac{e^{\lambda}-1}{r^2}=0 \qquad \dots \quad (10)$$

The equation (9) is satisfied by

$$e^{-\lambda}=1-\frac{A'}{r}, \qquad \dots \quad (11)$$

where A' is the constant of integration.

If the solution is to be regular at the origin, we must have $A'=0$, so that

$$e^{\lambda}=1 \qquad \dots \quad (12)$$

Substituting this in (10), we have

$$\frac{d^2\nu}{dr^2}+\frac{1}{2}\left(\frac{d\nu}{dr}\right)^2-\frac{1}{r}\frac{d\nu}{dr}=0 \qquad \dots \quad (13)$$

Multiplying by $e^{\frac{1}{2}\nu}$ and putting $e^{\frac{1}{2}\nu}=z$, the equation (13) reduces to

$$\frac{d^2z}{dr^2}-\frac{1}{r}\frac{dz}{dr}=0$$

so that

$$z=A+Br^2.$$

Hence

$$e^{\nu} = (A+Br^2)^2 \qquad \dots \quad (14)$$

From (3), (12), and (14), we get

$$\rho = p = -T_1^1 = \frac{1}{8\pi} e^{-\lambda} \left(\frac{1}{r}\frac{d\nu}{dr} - \frac{e^{\lambda}-1}{r^2} \right) = \frac{B}{2\pi} \frac{1}{A+Br^2} \qquad (15)$$

Thus, the field of an ideal fluid, defined by the relation $\rho = p$, is given by the metric

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + (A+Br^2)^2 dt^2, \qquad \dots \quad (16)$$

the pressure and density being given by

$$\frac{B}{2\pi(A+Br^2)}.$$

§ 5. *Case* (*a*).

Suppose that the gas is enclosed in a spherical envelope of radius a and the pressure on the envelope is P. We have $P = \frac{B}{2\pi(A+Ba^2)}$ whence $B = \frac{2\pi PA}{1-2\pi Pa^2}$. Also, if $P=0$, we have $B=0$ and therefore, from (15), $\rho = p = 0$ everywhere, so that the space is empty and so e^{ν} must be 1. This gives $A=1$. Hence we have

$$e^{\nu} = \left\{ 1 + \frac{2\pi Pr^2}{1-2\pi Pa^2} \right\}^2 \qquad \dots \quad (17)$$

and

$$p = \rho = \frac{P}{(1-2\pi a^2) + 2\pi Pr^2} \qquad \dots \quad (18)$$

The pressure at the origin $= \frac{P}{1-2\pi Pa^2}$. If this is to be positive and P is positive, we must have $P < \frac{1}{2\pi a^2}$ or $a^2 < \frac{1}{2\pi P}$. If $2\pi Pa^2 = 1$, e^{ν} is infinite. If $P < \frac{1}{2\pi a^2}$, let $2\pi a^2 P - 1 = 2\pi Pc^2$, so that

$$p = \rho = \frac{P}{2\pi P(r^2 - c^2)}, \; e^{\nu} = \left(1 + \frac{r^2}{c^2} \right)^2,$$

which shews that, in this case, the solution has a singularity at $r=c$ (which is less than a). Thus we see that, when $P < \frac{1}{2\pi a^2}$, the solution is regular and the pressure is positive everywhere.

Case (*b*).

If the space occupied by the fluid extends from $r=a$ to $r=\infty$, and if P be the pressure at $r=a$, we have, as before,

$$p=\rho=\frac{P}{1+2\pi P(r^2-a^2)}$$

Since $r>a$, the pressure is positive everywhere. It vanishes at $r=\infty$, where e^ν becomes infinite. e^ν is also infinite when $P=\frac{1}{2\pi a^2}$.

Thus, in this case, unless $P=\frac{1}{2\pi a^2}$ the solution is regular everywhere except at infinity.

Case (*c*).

Suppose now that the whole space is occupied by the fluid, the pressure at the origin being given to be p_0.

Putting $a=0$ in the solution of case (*b*), we have

$$p=\rho=\frac{p_0}{1+2\pi p_0 r^2} \quad \dots \quad (19)$$

$$e^\nu=(1+2\pi p_0 r^2)^2 \quad \dots \quad (20)$$

If p_0 is assumed to be positive, the pressure and density are positive everywhere and vanish at infinity.

The solution is regular everywhere except at infinity.

§6. The spatial part of the metric (16) represents the Euclidean three-space, while the Riemann-curvature of the space-time (16) is given by

$$\left.\begin{aligned} K&=8\pi T\\ &=-24\pi p\\ &=\frac{-24\pi P}{1-2\pi Pa^2+2\pi Pr^2} \text{ in cases } (a) \text{ and } (b)\\ \text{and} \quad &=\frac{-24p_0}{1+2\pi p_0 r^2} \text{ in case } (c) \end{aligned}\right\} \quad \dots \quad (21)$$

The curvature is therefore of opposite sign to the pressure and it vanishes at infinity.

§7. To determine the paths of light as affected by the gravitation of matter through which it passes, we observe that, if a ray of light starts in the plane $\theta=\frac{\pi}{2}$, it will continue to travel in the same plane, so that it will be sufficient to obtain the geodesic null lines in the plane $\theta=\frac{\pi}{2}$.

Putting r, θ, ϕ, $t=x_a$ $(a=1, 2, 3, 4)$, the geodesics of the space-time (17) are given by

$$\frac{d^2 x_a}{ds^2}+\{pq, a\}\frac{dx_p}{ds}\frac{dx_q}{ds}=0, \quad (a=1, 2, 3, 4) \qquad \ldots \quad (22)$$

where all the Christoffel symbols vanish except the following:

$$\left.\begin{array}{l} \{12, 2\}=\frac{1}{r}, \quad \{13, 3\}=\frac{1}{r}, \quad \{14, 4\}=\frac{2Br}{1+Br^2} \\ \{22, 1\}=-r, \quad \{33, 1\}=-r, \quad \{44, 1\}=2Br(1+Br^2) \end{array}\right\} \qquad (23)$$

We thus get

$$\frac{d^2 r}{ds^2}-r\left(\frac{d\phi}{ds}\right)^2+2Br(1+Br^2)\left(\frac{dt}{ds}\right)=0$$

$$0 \qquad =0$$

$$\frac{d^2\phi}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} \qquad =0 \qquad \ldots \quad (24)$$

$$\frac{d^2 t}{ds^2}+\frac{4Br}{1+Br^2}\frac{dr}{ds}\frac{dt}{ds} \qquad =0 \qquad \ldots \quad (25)$$

And from (16)

$$\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\phi}{ds}\right)^2-(1+Br^2)^2\left(\frac{dt}{ds}\right)^2+1=$$

From (24) and (25), we have

$$\left.\begin{aligned} r^2\frac{d\phi}{ds}&=h, \\ \frac{dt}{ds}&=\frac{\kappa}{(1+Br^2)^2} \end{aligned}\right\}$$

h, κ being constants.

Substituting these in (26), we have

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}-\frac{\kappa^2}{(1+Br^2)^2}+1=0$$

or, if $u=\frac{1}{r}$,

$$\left(\frac{du}{d\phi}\right)^2+u^2+\frac{1}{h^2}-\frac{k^2}{h^2}\frac{u^4}{(B+u^2)^2}=0,$$

whence, on differentiation,

$$\frac{d^2u}{d\phi^2}+u=\frac{k^2}{h^2}\cdot\frac{2Bu^3}{(B+u^2)^3}$$

which is the differential equation of the geodesics.

For null lines, we must have $ds=0$. Hence from (27) h be infinite, while $\frac{h}{\kappa}$ is finite. Hence the equation of the g lines can be written

$$\frac{d^2u}{d\phi^2}+u=\frac{m^2u^3}{(n^2+u^2)^3}$$

where m^2 is an arbitrary parameter and n^2 is put for assumed to be positive.

IV. *The field of an ideal fluid according to the modified law of gravitation.*

§8. We now take the first form of the equations (3)......(6). By equating (3) and (4) we obtain an equation which is satisfied by (10) and (11), *viz.*,

$$\frac{1}{2}\frac{d^2\nu}{dr^2}-\frac{1}{4}\frac{d\lambda}{dr}\frac{d\nu}{dr}+\frac{1}{4}\left(\frac{d\nu}{dr}\right)^2-\frac{1}{2r}\left(\frac{d\nu}{dr}\right)=0 \qquad \ldots \quad (10)$$

$$\frac{d\lambda}{dr}-\frac{2(1-e^{\lambda})}{r}=0 \qquad \ldots \quad (11)$$

Also from (6)

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\lambda}{dr}+\frac{e^{\lambda}-1}{r^2}\right)=\beta \qquad \ldots \quad (30)$$

The solution of (11) is

$$e^{\lambda}=(1+Cr^2)^{-1},$$

which satisfies (30) if $C=\frac{1}{3}\beta$. Hence

$$e^{\lambda}=(1-\tfrac{1}{3}\beta r^2)^{-1} \qquad \ldots \quad (31)$$

Substituting from (31) in (10), we have

$$\frac{1}{2}\frac{d^2\nu}{dr^2}+\frac{1}{4}\left(\frac{d\nu}{dr}\right)^2-\frac{1}{2r(1-\frac{1}{3}\beta r^2)}\frac{d\nu}{dr}=0.$$

Multiplying by $e^{\frac{1}{2}\nu}$ and putting $e^{\frac{1}{2}\nu}=z$, this becomes

$$\frac{d^2z}{dr^2}-\frac{1}{r(1-\frac{1}{3}\beta r^2)}\frac{dz}{dr}=0$$

which gives

$$z=A-B(1-\tfrac{1}{3}\beta r^2)^{\frac{1}{2}},$$

where A and B are constants of integration.

Hence

$$e^{\nu}=\{A-B(1-\tfrac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2 \qquad \ldots \quad (32)$$

From equation (1), we have

$$p=-T_1^1=\frac{1}{8\pi}\left[\beta+e^{-\lambda}\left(\frac{1}{r}\frac{d\nu}{dr}-\frac{e^{\lambda}-1}{r^2}\right)\right]$$

$$=\frac{\beta}{12\pi}\cdot\frac{A}{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}} \qquad \ldots \quad (33)$$

Thus, the field of a mass of ideal fluid is given by the metric

$$ds^2=-(1-\tfrac{1}{3}\beta r^2)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2$$

$$+\{A-B(1-\tfrac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2dt^2 \qquad \ldots \quad (34)$$

and the pressure at any point by

$$p=\beta A/12\pi\{A-B(1-\tfrac{1}{3}\beta r^2)^{\frac{1}{2}}\}.$$

§9. The solution (34) is of the same form as the solution of Schwarzschild for a medium which he defines as an incompressible liquid on the assumption that the cosmological constant is zero. But, as we have already mentioned in Art. 1, Schwarzschild's condition of compressibility is open to objection and so his form of solution really refers to the field, not of a liquid in a flat world, but of an ideal gas in the naturally curved world. This seems to be a more correct interpretation of the *form* of the solution obtained by Schwarzschild.

It should be observed, however, that, though the *form* of the solution is identical in the two cases, the properties of the field (*e.g.*, the distribution of pressure) are entirely different. We shall therefore proceed to discuss the solution briefly with reference to the medium under our consideration.

§10. *Case* (*a*).

Suppose that the external boundary of the fluid is a sphere of radius a under a given pressure P. We have from (33)

$$P=\frac{\beta}{12\pi}\cdot\frac{A}{A-B(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}}$$

If $\beta=0$, the curvature of space is zero and from (33) the pressure is zero everywhere, so that we get an empty galilean space. Hence e^{ν} must be equal to unity in this case. Thus from (32), $A-B=1$. We therefore have

$$A=\frac{P(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}}{P\{(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}-1\}+\kappa},$$

$$B=\frac{P-\kappa}{P\{(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}-1\}+\kappa}$$

where $\kappa=\frac{\beta}{12\pi}$.

From (33) we have

$$p=\frac{\kappa P(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}}{P(1-\frac{1}{3}\beta a^2)^{\frac{1}{2}}-(P-\kappa)(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}} \qquad \dots \quad (35)$$

The pressure will be finite everywhere unless

$$a^2 > \frac{3}{\beta}\left\{1-\left(1-\frac{\kappa}{P}\right)^2\right\} \text{ or } > \frac{1}{2\pi P}-\frac{\beta}{48\pi^2P^2}. \qquad \dots \quad (36)$$

This gives an upper limit to the size of the sphere of the fluid under a given pressure. If we neglect β, this limit is

$$a^2=\frac{1}{2\pi P} \quad [cf. \text{ Art. 5, case } (a)].$$

Case (*b*).

Suppose that the fluid fills all space and that the pressure at the origin is given to be p_0. We then have

$$A-B=1, \qquad p_0=\frac{\kappa A}{A-B},$$

so that

$$A=\frac{p_0}{\kappa}, \qquad B=\frac{p_0-\kappa}{\kappa} \qquad \dots \quad (37)$$

and

$$p=\frac{\kappa p_0}{p_0-(p_0-\kappa)(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}} \qquad \dots \quad (38)$$

The pressure is finite and positive everywhere.

Since $1-\frac{1}{3}\beta r^2$ is never negative, the limiting value of p is equal to $\kappa=\frac{\beta}{12\pi}$, and this depends only on the natural curvature of space and not on the value of the pressure at the origin.

§11. In case (*a*), if $P=0$, we find from (35), that $p=0$ everywhere, so that the space is empty. In this case, $A=0$, $B=-1$ and since $e^{\nu}=\{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2$, we get $e^{\nu}=1-\frac{1}{3}\beta r^2$, which is the solution for De Sitter's empty world.

In case (*b*), if $p_0=0$, we find $p=0$ everywhere from (38) and the space is empty. From (37), we have $A=0$, $B=-1$. Thus the solution reduces to De Sitter's space-time.

The spatial part of the metric is identical with that of the metric of De Sitter's world and the momentary 3-space ($dt=0$) is therefore a non-Euclidean hyperbolic space. It will be observed that the distribution of pressure and density at any point is bound up with the constant natural curvature of space and the presence of the fluid changes the constant curvature into a variable one.

If K be the Riemann-curvature, we have

$$K-4\beta=8\pi T$$

$$=\frac{-2\beta A}{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}}$$

or, $$K=4\beta\left\{1-\frac{\frac{1}{2}A}{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}}\right\}. \qquad \dots (39)$$

In case (*b*), *i.e.*, when the fluid fills all space, we have

$$K=4\beta-\frac{24\pi p_0\beta}{12\pi p_0-(12\pi p_0-\beta)(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}}.$$

If β^2 be neglected, we have approximately

$$K=4\beta-\frac{24\pi p_0}{1+2\pi p_0 r^2},$$

whence it follows that the curvature changes sign at

$$r=\sqrt{\frac{6\pi p_0-\beta}{2\pi p_0\beta}},$$

and if $p_0<\frac{\beta}{6\pi}$, the curvature will be of the same sign throughout.

§12. To obtain the paths of light as modified by the presence of gravitation of the fluid in De Sitter world, we shall find as before, the geodesic null lines in the plane $\theta=\frac{\pi}{2}$.

Proceeding as in Art. 7, the equations of the geodesics are found to be

$$\frac{d^2r}{ds^2}+\frac{\beta r}{3(1-\frac{1}{3}\beta r^2)}\left(\frac{dr}{ds}\right)^2-r(1-\frac{1}{3}\beta r^2)\left(\frac{d\phi}{ds}\right)^2$$

$$+\tfrac{1}{3}(1-\tfrac{1}{3}\beta r^2)\{A-B(1-\tfrac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2\left(\frac{dt}{ds}\right)=0$$

$$\frac{d^2\phi}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}=0 \qquad \ldots \quad (40)$$

$$\frac{d^2t}{ds^2}+\frac{2\beta Br}{3(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}\{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}\}}\frac{dr}{ds}\frac{dt}{ds}=0 \qquad \ldots \quad (41)$$

From (34) we have

$$\frac{1}{1-\frac{1}{3}\beta r^2}\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\phi}{ds}\right)^2-\{A-B(1-\tfrac{1}{3}\beta r^2)\}^2\left(\frac{dt}{ds}\right)^2+1=0 \quad \ldots \quad (42)$$

Integrating (40) and (41), we have

$$r^2\frac{d\phi}{ds}=h$$

$$\frac{dt}{ds}=\frac{c}{\{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2}$$

where h and c are constants of integration. Substituting these in (42) we have

$$\frac{1}{1-\frac{1}{3}\beta r^2}\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}-\frac{c^2}{\{A-B(1-\frac{1}{3}\beta r^2)^{\frac{1}{2}}\}^2}+1=0 \qquad \ldots \quad (43)$$

Since $ds=0$ for a null line, we must have h and c infinite while $\frac{c}{h}$ will remain finite. Hence from (43) we obtain

$$\frac{3u^2}{3u^2-\beta}\left(\frac{du}{d\phi}\right)^2+u^2=\frac{3m^2u^2}{\{uA\sqrt{3}-B(3u^2-\beta)^{\frac{1}{2}}\}^2}$$

where

$$u=\frac{1}{r},\ m^2=\frac{c^2}{h^2}.$$

If the squares and higher powers of β are neglected, we get, after some reduction,

$$\left(\frac{du}{d\phi}\right)^2=\frac{3m^2u^2(3u^2+\beta)-(A-B)u^2\{3(A-B)u^2+B\beta\}}{(A-B)(u^2+A\beta)}$$

which is of the form

$$\frac{du}{d\phi}=\pm\sqrt{\frac{u^2(Pu^2+Q)}{P'u^2+Q'}}.$$

V. *Conclusion and summary.*

§13. Following the procedure adopted by Schrödinger, Silberstein, Bauer and Eddington, we have assumed the material-energy-tensor for a perfect fluid to be given by (7), *viz.*,

$$T^{\nu}_{\mu}=-\delta^{\nu}_{\mu}\,p+g_{\mu\sigma}\,\frac{dx_{\nu}}{ds}\,\frac{dx_{\sigma}}{ds}\,\rho$$

where p and ρ are scalars representing the pressure and density of the fluid at any point. We then *define* a perfect fluid of *ideal* compressibility by the equation $p=\rho$.

The gravitational fields of such an ideal fluid have been found in the case when the gravitational equations are of the original form and also when they contain the cosmological constant. The curvatures of the spaces and the differential equations of the light-paths have been determined. The solutions reduce to the galilean and the De Sitter forms respectively when the energy-tensor vanishes.

It is proved that the form of solution which Schwarzschild interprets as belonging to the field of an incompressible liquid sphere placed in a flat space really represents the field of an ideal gas placed in the naturally curved world.

If a fluid sphere under external pressure P is placed in the galilean field the consequent curvature sets a limit to the radius a of the sphere,

the limit being given by $a^2 = \frac{1}{2\pi P}$. The corresponding limit in the case of the naturally curved world is given by

$$a^2 = \frac{3}{\beta}\left\{1 - \left(1 - \frac{\beta}{12\pi P}\right)^2\right\},$$

which reduces to $\frac{1}{2\pi P}$, if β is neglected, and thus agrees with the first case.

When the fluid fills all space, the pressure, in the first case, diminishes from the given value at the origin to zero at infinity, independently of the initial value. In the second case, the limiting value, reached at $r = \sqrt{\frac{3}{\beta}}$, is found to be $\frac{\beta}{12\pi}$, which depends on the natural curvature of space, and is also independent of the pressure at the origin.

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4

Table of Complex Multiplication Moduli of Elliptic Functions for Some New Cases*

BY

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(I) $\Delta=106$.

$$(kk'+\lambda\lambda')^2=2x^{12}(8y^6-24y^4+26y^2-10)(2y^3-8y^2+9y-2)^2,$$

where

$$4kk'\lambda\lambda'=x^{12}, \quad x+\frac{1}{x}=\sqrt{2}y,$$

$$2y^3-12y^2+23y-12=0,$$

and
$$kk'=i\,\frac{\phi^4(\sqrt{-106})}{\psi^8(\sqrt{-106})}.$$

(II) $\Delta=109$.

$$(4kk'\lambda\lambda')^{\frac{1}{12}}=x, \qquad x+\frac{1}{x}=\sqrt{2}y.$$

$$(kk'+\lambda\lambda')=(1-4x^2+4x^4-x^6)(1-4\sqrt{2}x+12x^2-10\sqrt{2}x^3$$
$$+12x^4-4\sqrt{2}x^5+x^6).$$

and
$$2y^3-4y^2+9y-9=0.$$

* With the exception of (IV), XI, XIII and XIV, the other cases are *entirely* new; for these cases Dr. Berwick has given the class invariants but not the moduli.

(III) $\Delta=517$.

$$(4kk'\lambda\lambda')^{\frac{1}{12}}=x, \quad x+\frac{1}{x}=\sqrt{2y},$$

$$kk'+\lambda\lambda'=\sqrt{2}(1-4x^2+4x^4-x^6)\,(2x-3x^3+2x^5)$$

and
$$2y^3-20y^2-71y-55=0.$$

(IV) $\Delta=235$.

$$(4k\lambda k'\lambda')^{\frac{1}{12}}=x, \qquad y=\sqrt{2}x.$$

$$\sqrt[4]{2kk'}=\left(\frac{4y+2y^3-y^5}{8\sqrt{2}}\right)^{\frac{1}{2}}-\left(\frac{4y-2y^3-y^5}{8\sqrt{2}}\right)^{\frac{1}{2}}$$

and

$$y=-2+\left(3+\frac{\sqrt{235}}{3\sqrt{3}}\right)^{\frac{1}{3}}+\left(3-\frac{\sqrt{235}}{3\sqrt{3}}\right)^{\frac{1}{3}}.$$

(V) $\Delta=355$.

$$\sqrt[4]{2kk'}=\left(\frac{4y+2y^3-y^5}{8\sqrt{2}}\right)^{\frac{1}{2}}-\left(\frac{4y-2y^3-y^5}{8\sqrt{2}}\right)^{\frac{1}{2}}.$$

$$y=-\frac{(\sqrt{5}+1)}{3}+\frac{1}{3}\left\{(55\sqrt{5}+146)+(6+3\sqrt{5})\,(1065)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

$$+\frac{1}{3}\left\{(55\sqrt{5}+146)-(6+3\sqrt{5})\,(1065)^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

(VI) $\Delta=529$.

$$x+\frac{1}{x}=\sqrt{2}y, \qquad x=(2kk')^{\frac{1}{12}}.$$

$$(2y^3-16y^2+12y-3)^2-23(2y^2-y)^2=0.$$

(VII) $\Delta=89$.

$$(kk'\lambda\lambda')^{\frac{1}{4}}=x.$$

$$\sqrt{2}x+\frac{1}{\sqrt{2x}}=\sqrt{2}\,zi,$$

$$z^3-(17-32i)z^2-(61+16i)z-(11+16i)=0.$$

$$\sqrt[4]{k}=e^{\pi i/4}\,\phi(\sqrt{-89}),\qquad \sqrt[4]{k'}=\psi(\sqrt{-89}).$$

(VIII) $\Delta=171$.

$$J=-\frac{(\tau-1)\ (9\tau-1)^3}{64\tau}.$$

$$\tau=(8199+1086\sqrt{57})+(135+18\sqrt{57})^{\frac{1}{2}}\,(498+66\sqrt{57}).$$

(IX) $\Delta=387$.

$$\tau=(8192007+721266\sqrt{129})+(459378+40446\sqrt{129})$$
$$\times(159+14\sqrt{129})^{\frac{1}{2}}.$$

(X) $\Delta=603$.

$$27(\tau-1)=8v^3.$$

$$2v=(440+31\sqrt{201})+3(43071+3038\sqrt{201})^{\frac{1}{2}}.$$

(XI) $\Delta=243$.

$$\tau=\left(1-\frac{2}{3^{1/3}}\right)^{-4}.$$

(XII) $\Delta=175$.

$$x=\sqrt[6]{16kk'}.$$

$$(2x^3-x^2+11x-7)^2-45(x^2-x+1)^2=0.$$

(XIII) $\Delta=123$.

$$t=\sqrt[4]{4kk'},$$

$$(t^3-4t^2+20t-1)^2-41(t^3-3t)^2=0.$$

(XIV) $\Delta=139$.

$$a=\frac{(1-16k^2k'^2)^3}{16k^2k'^2}.$$

$$\left(\frac{a}{2}\right)-28764\left(\frac{a}{2}\right)^{\frac{2}{3}}+36720\left(\frac{a}{2}\right)^{\frac{1}{3}}-79488=0.$$

(XV) $\Delta=155$.

$$\left(\frac{a}{2}\right)^{\frac{4}{3}}-57420\left(\frac{a}{2}\right)+280720\left(\frac{a}{2}\right)^{\frac{2}{3}}-5638400\left(\frac{a}{2}\right)^{\frac{1}{3}}$$

$$+8166400=0.$$

(XVI) $\Delta=118$.

$$y=-\sqrt{2}\cosh 2\phi,$$

$$y^3+35y^2-21y+9=0.$$

The proofs will appear in a subsequent paper

5

The Hindu solution of the general Pellian Equation

BY

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Introductory.

It is well known that the Hindus had the very remarkable success in obtaining the general solution in *rational integers* of the indeterminate equations

$$ax^2+1=y^2, \qquad \ldots \quad (1)$$

$$ax^2+c=y^2, \qquad \ldots \quad (2)$$

where a, c are *non-square* integers. Of their method of solution, which is called the *Cakravāla* (or the "cyclic method") and whereby the Hindus long anticipated the works of Lagrange (1768), Hankel observed: "It is above all praise; it is certainly the finest thing which was achieved in the theory of numbers before Lagrange." The credit for the solution, so far as known, is due to Brahmagupta (628) and Bhāskara (1150). As the algebra of Śrīdhara (*c.* 750) and of Padmanābha, upon which Bhāskara, according to his own admission, had drawn heavily, are lost to us, it cannot be ascertained now how much of the credit in this matter, is due to the either of them. A scholiast of Bhāskara, Sūryadāsa (1540), thinks that he was also influenced by the writings of Prithudakasvāmī (860). There are good grounds for such a supposition. For instance, in his treatment of the *Vargaprakṛti* (equivalent to our "Pellian equations"), Bhāskara has adopted certain technical terms which can be traced back to Pridudakasvāmī.*

* H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara*, London, 1817, p. 363, footnote; hereafter referred to as Colebrooke, *Hindu Algebra.*

These terms are more noteworthy as being not always appropriate.* Still they have been adopted by all the subsequent writers. The corresponding terms of Brahmagupta are, indeed, faultless.

The object of the present note is to point out that the Hindus also accomplished the complete solution of the more general equation

$$ax^2+bx+c=y^2 \qquad \dots \quad (3)$$

Hence in this case too they anticipated Euler and Lagrange. This fact has been overlooked by the modern historians of mathematics. It is not mentioned even in Professor Dickson's monumental work on the history of the theory of numbers.† Colebrooke's imperfect translation is no less responsible for this neglect.

Hindu Solution.

The earliest mention of the solution of $ax^2+bx+c=y^2$ is now met with in the *Bījagaṇita* of Bhāskara (1150).‡ The method is a very simple and elegant one. It consists in "completing the square" on the left-hand side of the equation, so that it is at once transformed into another of the known form (2). Thus

$$ax^2+bx+c=y^2,$$

or

$$\left(ax+\frac{b}{2}\right)^2=ay^2+\frac{1}{4}(b^2-4ac).$$

Putting $z=ax+\dfrac{b}{2}$, $\alpha=\dfrac{1}{4}(b^2-4ac)$, we get

$$ay^2+\alpha=z^2 \qquad (3.1)$$

* If $x=m$, $y=n$ be a solution of the equation

$$ax^2+c=y^2,$$

Bhāskara calls m *hrasva mūla* ("the lesser root") and n *jyeṣṭa mūla* ("the greater root"). These terms will have true and proper significance, if c is positive; but in case of its being negative, the reverse will be sometimes more appropriate. Brahmagupta used the terms *ādya mūla* ("first root") and *antya mūla* ("last root"). Compare Colebrooke, *Hindu Algebra*, pp. 170, 363; *Brāhma-sphuṭa-siddhānta*, xviii. 64.

† L. E. Dickson, *History of the Theory of Numbers*, Vols. I. & II, Washington, 1920.

‡ Colebrooke, *Hindu Algebra*, p. 245 *et seq.*

To complete the solution : if $y=l$, $z=m$ be found tentatively to be a solution of this equation, another solution of it is known to be

$$y=lq\pm mp, \qquad z=mq\pm alp;$$

where $\quad q^2=ap^2+1$. Hence a solution of (3) will be

$$x=-\frac{b}{2a}+\frac{1}{a}(mq\pm alp),$$

$$y=lq\pm mp.$$

Now suppose $x=r$, when $z=m$; that is, let $m=ar+b/2$. Substituting in the above expressions, we get the required solution of (3) as

$$x=\frac{1}{2a}(bq-b)+qr\pm lp, \qquad \dots \quad (3.2)$$

$$y=lq\pm(apr+\frac{1}{2}bp);$$

where $\quad q^2=ap^2+1$ and $ar^2+br+c=l^2$.

Thus having known one solution of $ax^2+bx+c=y^2$, we can easily obtain an infinite number of other solutions by the method indicated by Bhāskara.

Euler's Solution : Lagrange.

The above solution (3.2), but with the positive sign only, was rediscovered in 1733 by Euler.* His method is indirect and, moreover, cumbrous. It is further akin to that of Diophantus (VI. 15, Lemma). Euler starts with assuming the values

$$x=\alpha r+\gamma l+\beta,$$

$$y=\delta r+\zeta l+\epsilon;$$

where $x=r$, $y=l$ is a previously known solution of the equation (3) and

* Leonardi Euleri, *Opera Mathematica*, Vol. II, 1915, pp. 6-17; compare also pp. 576-611 for other methods of Euler.

$\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are undetermined quantities. Substituting in the equation and since $l^2=ar^2+br+c$, equating the al and the co-efficients of different powers of r on either equation, the following equations of condition are obtained:

$$a\alpha^2+a^2\gamma^2=\delta^2+a\zeta^2,$$

$$2a\alpha\beta+ab\gamma^2+b\alpha=2\delta\epsilon+b\zeta,$$

$$a\beta^2+ac\gamma^2+b\beta+c=\epsilon^2+c\zeta^2,$$

$$2a\alpha\gamma=2\delta\epsilon.$$

$$2a\beta\gamma+b\gamma=2\epsilon\zeta.$$

From these Euler succeeds in evaluating the undetermin such that

$$\alpha=q, \quad \gamma=p, \quad \beta=\frac{b}{2a}(q-1),$$

$$\delta=ap, \quad \epsilon=\frac{1}{2}bp, \quad \zeta=q;$$

where $q^2=ap^2+1$. Hence

$$x=qr+pl+\frac{b}{2a}(q-1);$$

$$y=apr+ql+\tfrac{1}{2}bp.$$

Lagrange begins in the same way as Bhāskara by con square on the left-hand side.*

* Additions to *Elements of Algebra*, by Leonard Euler, translated by John Howlett, 5th edition, London, 1840, p. 587.

Greek Solution.

The Greek Diophantus obtained solution of some particular cases of $ax^2+bx+c=y^2$. According to him a rational solution of it is possible only where (*i*) a is positive and a square, (*ii*) c is positive and a square, or when (*iii*) $\frac{1}{4}b^2-ac$ is positive and a square number. His method of solution is entirely different from that of the Hindus and consists of assuming a suitable value for y so as to make the equation simple and determinate. Thus in case (*i*), he sets $y=ax-m$ and in case (*ii*), he sets $y=mx-c$. The case (*iii*) has not been expressly enunciated by Diophantus, but it occurs incidentally in an example (IV. 31).*

Discoverer of the Hindu solution.

It has been stated before that the earliest mention of the Hindu solution of the general Pellian equation is now found in the *Bījagaṇita* of Bhāskara (1150). But there are materials to suggest that he was not its first discoverer. He has taken a few illustrative examples from certain earlier writers whose solution presupposes a knowledge of the solution of the general Pellian equation. Neither those illustrations, nor treatment of equations of those types occur in the algebra of Brahmagupta or in any other known work anterior to Bhāskara. One of the illustrations runs thus† :—

"The square of the sum of too numbers, added to the cube of their sum, is equal to twice the sum of their cubes. Tell the numbers, mathematician !"

If x, y be the numbers, then

$$(x+y)^2+(x+y)^3=2\ (x^3+y^3).$$

The method of solution proposed is to put $x=u+v$, $y=u-v$, so that the equation will then become

$$4\,u^3+4u^2=12\ uv^2,$$

$$\text{or}\qquad 4\,u^2+4u=12\ v^2,$$

$$\text{or}\qquad (2u+1)^2=12v^2+1.$$

* T. L. Heath, *Diophantus of Alexandria,* 2nd edition, 1910, Cambridge, pp. 70-71.

† Colebrooke, *Hindu Algebra,* p. 248.

Another example* culled from a different writer leads to the equation

$$\sqrt[3]{\frac{xy+y}{2}}+\sqrt{x^2+y^2}+\sqrt{x^2-y^2+8}+\sqrt{x+y+2}+\sqrt{x-y+2}=z^2.$$

There have been indicated several ingenious hints for solving this equation.

(*i*) Set $x=u^2-1$, $y=2u$; then the equation becomes

$$2u^2+3u-2=z^2.$$

(*ii*) Set $x=v^2+2v$, $y=2v+2$; then

$$2v^2+7v+3=z^2,$$

(*iii*) Set $x=v^2-2v$, $y=2v-2$; then

$$2v^2-v-3=z^2,$$

(*iv*) Or set $x=v^2+4v+3$, $y=2v+4$; in this case the equation will be reduced to

$$2v^2+11v+12=z^2.$$

So that in every way, the solution of the equation obtained depends upon the solution of a Pellian equation of the general form.

A third example† of an earlier writer consists of two double equations of the second degree:

$$(1)\quad x^2+y^2+1=u^2,$$

$$x^2-y^2+1=v^2\ ;$$

$$(2)\quad x^2+y^2-1=u^2$$

$$x^2-y^2-1=v^2.$$

* Colebrooke, *Hindu Algebra*, p. 255.

† *Ibid*, p. 257.

The above illustrations will prove it undoubtedly that those earlier writers who have been referred to by Bhāskara must have been well acquainted with the solution of the general Pellian equation. It is to be regretted for the history of Hindu mathematics that their names and works have still remained sealed to us.

Solution of $ax^2+bx+c=a'y^2+b'y+c'$.

Bhāskara treated a still more general type of intermediate equations $ax^2+bx+c=a'y^2+b'y+c'$. The method of solution is the same as in the previous case. Completing the square on one side, say on the left, the equation reduces to another

$$aa'y^2+ab'y+\alpha=z^2$$

where $z=ax+\frac{1}{2}b$

$$\alpha=ac'+\tfrac{1}{4}b^2-ac.$$

The reduced equation can obviously be solved by the method stated before.*

As an illustration of the above method, Bhāskara works out an interesting example:† To find the number of terms of two series in A.P. whose first term is 3 and the common difference 2, but the sum of the one of which is 3 times the sum of the other.

If x and y denote the number of terms of the two series in A. P., then, by the question, we shall have to solve

$$3\times\frac{x}{2}\{3\times2+(x-1)2\}=\frac{y}{2}\{3\times2+(y-1)2\},$$

or $3x^2+6x=y^2+2y.$

Multiplying both sides by 3 and completing the square on the left hand side, we get

$$(3x+3)^2=3y^2+6y+9.$$

* Colebrooke, *Hindu Algebra*, p. 250.

† *Ibid*, p. 351 ; Colebrooke's translation of this stanza is not cear.

Setting $z=3x+3$, we have

$$3y^2+6y+9=z^2.$$

Again completing the square on the left

$$(3y+3)^2=3z^2-18.$$

Whence by the method of the affected square we can obtain an infinite number of solutions, such as

$$\left.\begin{array}{r} z=9 \\ 3y+3=15 \end{array}\right\}, \quad \left.\begin{array}{r} z=33 \\ 3y+3=57 \end{array}\right\}, \text{ etc.}$$

Hence finally $x=2$, $y=4$ or $x=10$, $y=18$ and so on.

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6

ON THE SUMMABILITY (C1) OF THE DERIVED SERIES OF THE FOURIER SERIES OF AN INDEFINITE INTEGRAL AT A POINT WHERE THE INTEGRAND HAS A DISCONTINUITY OF THE SECOND KIND

BY

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The object of the present paper is to obtain, with the help of the Infinitärcalcül of Du Bois-Reymond, as complete an answer as possible to the following question: What are the types of functions to which $f(x)$ must belong in order that the derived series of the Fourier series of the indefinite integral of $f(x)$ should be summable (C1) at a point x_0 where $f(x)$ has a discontinuity of the second kind? For this purpose, I use the results of a recent paper * of mine. Among other results, I find that there are cases of $f(x)$ in which the derived series is summable (C1) at x_0 even when the generalised differential coefficient † of the indefinite integral of $f(x)$ is *non-existent* at x_0.

The aforesaid question has not been attacked *directly* by any previous writer. But two interesting papers ‡ of Prof. W. H. Young throw much light on the question; and by using the Infinitärcalcül it can be shown that, in so far as my results and Prof. Young's overlap, they agree.

* "On the summability (C1) of the Fourier series of a function at a point where the function has an infinite discontinuity of the second kind." (This issue of the *Bulletin*, pp. 51-58.)

† By the generalized differential co-efficient of $\theta(x)$ at x_0, is, after Young, understood $\lim\limits_{h=0} \frac{\theta(x_0+h)-\theta(x_0-h)}{2h}$.

‡ On the usual convergence of a class of trigonometrical series." (*Proc. L.M.S*, Series 2, Vol. 13, 1914); "On the convergence of the derived series of Fourier series." (*Proc. L.M.S.*, Series 2. Vol. 17, 1918).

Throughout this paper, the indefinite integral of $f(x)$ is denoted by $F(x)$ and $\chi(t)$ and $\psi(t)$ are assumed to be functions which are each both monotone and unbounded in the neighbourhood of $t=0$.

1. It is easily seen that the Fourier series of

$$G(x) \equiv F(x) - \frac{1}{2}a_0 x - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, dt$$

is

$$\sum_1^\infty \frac{1}{n}\left(a_n \sin nx - b_n \cos nx\right),$$

where the a's and b's are the Fourier co-efficients of $f(x)$. Thus the derived series of the Fourier series of $F(x)$ is

$$\frac{1}{2}a_0\, T + \sum_1^\infty \left(a_n \cos nx + b_n \sin nx\right),$$

where T is the derived series of the Fourier series of x and is, therefore, * summable to 1; consequently, the question under consideration reduces to the question of the summability of the Fourier series of $f(x)$ at x_0.

2. Using $\phi(t)$ to denote $f(x_0+2t)+f(x_0-2t)$, let us consider one by one the following possibilities :—

(A) $\phi(t) = \cos \psi(t),\ \psi \succ \log \frac{1}{t^2}$;

(B) $\phi(t) = \cos \psi(t),\ \psi \preceq \log \frac{1}{t^2}$;

(C) $\phi(t) = \chi(t) \cos \psi(t),\ \psi \preceq \log \frac{1}{t^2}$

$$\left[\text{and, consequently, } \frac{\chi}{t\psi'} \succ 1\right];$$

* See the theorem given by Fejér relating to the derived series of the Fourier series of a function which is continuous and has a continuous differential co-efficient (*Math. Ann.*, *Bd.* 58, p. 61).

(D) $\phi(t)=\chi(t)\cos\psi(t)$, $\frac{\chi}{t\psi'} \precsim 1$

$$\left[\text{and, consequently, } \psi \succ \log \frac{1}{t^2}\right];$$

(E) $\phi(t)=\chi(t)\cos\psi(t)$, $\psi \succ \log\frac{1}{t^2}$, and $\frac{\chi}{t\psi'} \succ 1$ but $\frac{\chi}{(t\psi')^2} \precsim 1$

(F) $\phi(t)=\chi(t)\cos\psi(t)$, $\psi \succ \log\frac{1}{t^2}$, and $\frac{\chi}{(t\psi')^2} \succ 1$

but $\frac{\chi}{(t\psi')^3} \precsim 1$;

(G) $\phi(t)=\chi(t)\cos\psi(t)$, $\psi \succ \log\frac{1}{t^2}$, and $\frac{\chi}{(t\psi')^m} \succ 1$

but $\frac{\chi}{(t\psi')^{m+1}} \precsim 1$,

m being any integer greater than 1.

The results given in my paper, referred to above, enable us to settle the question of the summability of the derived series in every case excepting, perhaps, the last case. I proceed to state the results in each case.

3. (A) The derived series is summable at x_0 to the generalized differential co-efficient of F (x) at x_0 which exists.*

(B) and (C) The derived series is not summable at x_0 and the generalized differential co-efficient is non-existent at x_0.

(D) The derived series is summable at x_0 whether

$$\frac{\chi}{t\psi'} \prec 1 \text{ or } \frac{\chi}{t\psi'} \sim 1$$

although in the former case the generalized differential co-efficient is existent and in the latter case non-existent at x_0.

* The question of the existence of the differential co-efficient of F (x) is settled by using the results of my various papers on the subject. (See this *Bulletin*, Vols. 15 and 16 ; also *Proc. Benares M.S.*, 1926.)

(E) The generalized differential co-efficient is always non-existent at x_0, but the derived series is summable or not according as

$$\frac{\chi}{t\psi'} \prec t\sqrt{\psi''}$$

or

$$\frac{\chi}{t\psi'} \succ t\sqrt{\psi''}.$$

(F) The generalized differential co-efficient is always non-existent at x_0. and the derived series is definitely known to be not summable if

$$\frac{\chi}{(t\psi')^2} \prec t\sqrt{\psi''}.$$

(G) The generalized differential co-efficient is always non-existent at x_0, but there is uncertainty about the summability of the derived series on account of the insufficiency of our knowledge of oscillating Dirichlet's integrals.

It will be noticed that, under (D) and (E), we have definitely cases of summability with the non-existence of the generalized differential co-efficient.

4. I conclude this paper by stating Young's results and their bearing on the question of this paper.

(*a*) Probably the chief result of Young is the following theorem* of his second paper: The derived series of the Fourier series of F (x) converges (C1) at x_0 if the Fourier series of

$$\operatorname{cosec} u \int_0^u \{f(x_0+2t)+f(x_0-2t)\}\,dt$$

converges for $u=0$, *i.e.*, if the Fourier series of

$$\phi_1(u) \equiv \frac{1}{u}\int_0^u \phi(t)\,dt$$

converges for $u=0$.

* See p. 228, l. c.

(*i*) Applying the above test to (A), we find that the Fourier series converges ; for, $\phi_1(u)$ behaves as $\frac{1}{u\psi'(u)}$ $\sin\psi(u)$ and $\frac{1}{u\psi'(u)} \prec 1$.

(*ii*) Taking up (B), we find that $\phi_1(u)$ behaves respectively as $\cos\psi(u)$ or $\cos(\psi+\text{constant})$ according as $\psi \prec \log\frac{1}{t^2}$ or $\psi \sim \log\frac{1}{t^2}$; thus by Du Bois-Reymond's result* the Fourier series of $\phi_1(u)$ diverges for $u=0$ in each case.

(*iii*) For (C), it is seen that $\phi_1(u)$ behaves as

$$\chi_1(u)\cos\psi(u) \text{ or } \chi_1(u)\cos(\psi+\text{constant}),$$

where $\chi_1 \succ 1$; therefore by Du Bois-Reymond's result the Fourier series of $\phi_1(u)$ diverges for $u=0$.

(*iv*) In the case of (D), $\phi_1(u)$ behaves as $\frac{\chi(u)}{u\psi'(u)}\sin\psi(u)$; therefore by Du Bois-Reymond's result the Fourier series of $\phi_1(u)$ converges for $u=0$.

(*v*) For (E), $\phi_1(u)$ behaves as $\frac{\chi(u)}{u\psi'(u)}\sin\psi(u)$; therefore by Du Bois-Reymond's result the Fourier series of $\phi_1(u)$ is convergent or not according as

$$\frac{\chi}{t\psi'} \prec t\sqrt{\psi''} \text{ or } \frac{\chi}{t\psi'} \succ t\sqrt{\psi''}.$$

(*vi*) For (F) and (G), $\phi_1(u)$ behaves as $\frac{\chi(u)}{u\psi'(u)}\sin\psi(u)$; hence by Du Bois-Reymond's result the Fourier series diverges since

$$\frac{\chi}{\{u\psi'(u)\}^2} \succ 1.$$

* Du Bois-Reymond's result is that ρ being a monotone function

$$\int_0^a \rho(a)\cos\psi(a).\frac{\sin(2n+1)a}{a}\,da$$ tends to a limit with increasing n or does not

according as $\rho \prec a\sqrt{\psi''(a)}$ or $\succ a\sqrt{\psi''(a)}$. (See his memoir in *Abhandlungen d. b. Akademie der Wissenschaften*, 1876, p. 87.)

(*b*) The main result of Young's earlier paper for the purposes of the question of the present paper is the following: * The derived series of the Fourier series of a function of bounded variation converges almost everywhere to the differential co-efficient of the function.

It is not necessary to examine the above test at any length; it fails when $F'(x_0)$ is non-existent, or $F(x)$ is not of bounded variation in the neighbourhood of x_0, *e.g.* when $x_0=0$, $F(x)=x^2 \sin \frac{1}{x^2}$ for $x \geq 0$, and equal to $-x^2 \sin \frac{1}{x^2}$ for ≤ 0.

* See p. 21, l. c.

† In this case, $\phi(t)=2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}$, and this case falls under (D).

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THE THEORY OF ASSOCIATED FIGURES IN HYPERBOLIC GEOMETRY

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Introduction.

As early as 1829 Lobatschewsky in his work 'On the Principles of Geometry' observes that to every right-angled triangle on the Hyperbolic Plane there corresponds another, whose elements are completely determinate when those of the first are known.* He also notes the existence of a tri-rectangular quadrilateral associated to every right-angled triangle.† Lobatschewsky's proof of the latter correspondence was in part based on the use of Trigonometrical Formulæ. Engel established this correspondence from purely geometrical considerations using space of three dimensions.‡ Liebmann first obtained a proof of these correspondences independent both of Trigonometry and three dimensional space.§ From Lobatschewsky's correspondence between two right-angled triangles it is easy to deduce that any right-angled triangle always forms one of a set of five associated right-angled triangles.‖ The existence of five tri-rectangular quadrilaterals associated to these naturally follows. S. Mukhopadhyaya completed the

* Nikolai Jwanwitsch Lobaschfeskij, 'Zwei Geometrische Abhandlungen uebersetzt mit Anmerkungen Von Friedrich Engel', 2 Teile (Leipzig, 1898-99) p. 15 and p. 242.

† *Ibid*, p. 25 and p. 256.

‡ Engel, 'Zur nichteuklidischen Geometrie', *Leipsig. Ber. Ges. Wiss. Math. Phy. Klasse*, Vol. 50, p. 181 (1898).

§ Liebmann, 'Elementargeometrischer Beweis der Parallel-konstruktion und neue Begrundung den trignometrischen Formeln der hyperbolischen Geometrie,' *Math. Ann.* (Leipzig), Vol. 61, p 185 (1905).

‖ *Cf.* Sommerville, 'Non-Euclidean Geometry,' § 86, p. 74.

system of *associated Right-angled Triangles* (*with proper, improper or ideal vertices*) by adding an eleventh associated figure, namely a *rectangular pentagon*, and gave for the first time a synthetic proof of all the correspondences.*

In the present paper there are proved two fundamental theorems (§ 1) which have been used to give a simple synthetic proof of the correspondences mentioned above (§ 2) and to establish a transformation carrying over any *Triangle* (*with proper, improper or ideal vertices*) into an *associated Triangle* (§ 3). This transformation enables us to exhibit the known correspondences between the eleven Right-angled Triangles as particular cases of correspondences between Triangles not necessarily right-angled (§ 4). A complete solution of the problem of constructing a Triangle with the three angles (or the three sides) given also follows as a direct application of this transformation (§ 5). The problem has already been treated by various European writers and Liebmann has, in particular, devoted a comprehensive paper to it.† It will however be found that the method used in the present paper greatly simplifies the solution of the problem, by introducing a uniformity in the treatment of the various cases that arise.

I take this opportunity to thank Professor S. Mukhopadhyaya under whose guidance the investigations, the results of which are embodied here, were carried out.

§ 1. *Two Fundamental Theorems.*

Notation. Ordinary points—*proper points*—shall be denoted by usual capital letters, *e.g.*, A, P, while *improper points—points at infinity*—shall be denoted by Greek capitals, *e.g.*, Ω, Σ. The small letters p, q, r, s, u, v, w shall be used to denote proper straight lines.

The letters a, b, c, d, l, m, n stand for segments (of straight lines), all congruent segments being denoted by the same symbol. The

* S. Mukhopadhyaya, 'Geometrical Investigations on the correspondences between a right-angled triangle, a three-right-angled quadrilateral, and a rectangular pentagon in Hyperbolic Geometry,' *Bull. Cal. Math. Soc.*, Vol. 13, No. 4, p. 211 (1922-23).

† Leibmann, 'Die Konstruktion des Geradlinigen Dreiecks der nichteuklidischen Geometrie aus den drei Winkeln, *Ges. Wis. Math. Phys. Klasse.*, Vol. 53, p. 215 (1901).

Simon, 'Zwei Satze Zur nichteuklidischen Geometrie,' *Math. Ann.* (Leipzig), Vol. 48, p. 607 (1897) and 'Uber Dreieckskonstruktionen in der nichteuklidischen Geometrie,' *Math. Ann.* (Leipzig), Vol. 61, p. 587 (1905).

Grossmann, 'Die Konstruktion des Geradlinigen Dreiecks der nichteuklidischen Geometrie aus den drei Winkeln,' *Math. Ann.* (Leipzig), Vol. 58, p. 578 (1906).

corresponding angles of parallelism shall be denoted by $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ respectively. The segment complementary to a shall be denoted by a', and that complementary to AB by $(AB)'$. The functional symbols π and Δ shall have their usual meanings, *viz.*, $\pi(a)=\alpha$, $\Delta(\alpha)=a$.

Def. A triangle with all its angular points improper (at infinity) will be called a *limiting triangle.*

We proceed to state and prove two fundamental theorems regarding limiting triangles.

1 THEOREM I. *Suppose a line p, perpendicular at* P *to the side* $\Omega_1\Omega_2$ *of a limiting triangle* $\Omega_1\Omega_2\Omega_3$, *passes through a point* L_1 *on* $\Omega_1\Omega_3$, *and* L_2 *is the foot of the perpendicular from* L_1 *to* $\Omega_2\Omega_3$.

(*a*) *If q be a line perpendicular to p at a point* A *on it, and meeting* $\Omega_1\Omega_3$ *and* $\Omega_2\Omega_3$ *at* Q_1 *and* Q_2 *respectively.* [See Fig. (1)], *then*

$$AQ_1=\Delta(\angle L_2Q_2A), \quad AQ_2=\Delta(\angle L_1Q_1A); \text{ and } L_1Q_1=L_2Q_2.$$

(*b*) *If r be a line perpendicular to p at a point* B *on it, and possessing common perpendiculars* K_1R_1 *and* K_2R_2 *with* $\Omega_1\Omega_3$ *and* $\Omega_2\Omega_3$ *respectively* [See Fig. (2)], then

$$BK_1=(K_2R_2)', \quad BK_2=(K_1R_1)' \text{ and } L_1R_1=L_2R_2.$$

(*c*) *If the perpediculars from a point* C *on p to* $\Omega_1\Omega_3$ *and* $\Omega_2\Omega_3$ *meet these lines at* S_1 *and* S_2 *respectively* [See Fig. (3)], *then*

$$\angle L_1CS_1=\pi(CS_2), \quad \angle L_1CS_2=\pi(CS_1), \text{ and } L_1S_1=L_2S_2,$$

It is easy to see that the angle PL_1L_2 is a right angle and consequently L_1L_2 is the common perpendicular to $L_1P(p)$ and $\Omega_2\Omega_3$ [Fig. (1)]. Also drawing $P_1\Omega_3$ parallel to $\Omega_2\Omega_3$ and perpendicular to p at P_1 it is evident that lines perpendicular to p at points between P and P_1, meet both $\Omega_1\Omega_2$ and $\Omega_2\Omega_3$, while lines perpendicular to p at points outside PP_1 possess common perpendiculars with both $\Omega_1\Omega_3$ and $\Omega_2\Omega_3$.

Proof of (*a*). Draw the lines $F_1\Omega_1$, $F_2\Omega_2$, $F_3\Omega_3$ perpendicular to q at F_1, F_2, F_3 respectively [See Fig. (4)].

Now, $AF_1=AF_2$ each being complementary to AP,

or, $F_1Q_1-AQ_1=AQ_2-Q_2F_2$,

or, $\Delta(\angle L_1Q_1A)-AQ_1=AQ_2-\Delta(\angle L_2Q_2A)$.

Again, $Q_1F_3 + AQ_1 = AQ_2 + Q_2F_3$,

or, $\Delta(\angle L_1Q_1A) + AQ_1 = AQ_2 + \Delta(\angle L_2Q_2A)$.

Hence, $AQ_1 = \Delta(\angle L_2Q_2A)$ and $AQ_2 = \Delta(\angle L_1Q_1A)$.

Fig. (4) has been drawn for the case when A lies between L_1 and P_1. In the alternative when A lies between P and L_1 we have only to interchange the positive and negative signs in the above proof.

Draw $Q_1\Omega$, $Q_2\Omega$, $L_2\Omega$ parallel to $L_1A(PL_1)$ [See Fig. (5)].

Then, $\angle L_1Q_1A = \pi(AQ_2) = \angle \Omega Q_2A$,

$\angle \Omega Q_1A = \pi(AQ_1) = \angle L_2Q_2A$.

Therefore $\angle L_1Q_1\Omega = \angle L_2Q_2\Omega$. Also $\angle \Omega L_1Q_1 = \angle \Omega L_2Q_2$. Hence the infinite triangles $L_1Q_1\Omega$ and $L_2Q_2\Omega$ are congruent* and

$$L_1Q_1 = L_2Q_2.$$

Proof of (*b*). Draw the lines $G_1\Omega_1$, $G_2\Omega_2$, $G_3\Omega_3$ perpendicular to r at G_1, G_2, G_3, respectively [See Fig. (6)].

Now, $BG_1 = BG_2$, each being complementary to BP,

or, $K_1G_1 + BK_1 = BK_2 + K_2G_2$,

or, $(K_1R_1)' + BK_1 = BK_2 + (K_2R_2)'$

Again, $G_3K_2 - BK_1 = BK_2 - G_3K_1$,

or, $(K_1R_1)' - BK_1 = BK_2 - (K_2R_2)'$,

Hence, $BK_1 = (K_2R_2)'$ and $BK_2 = (K_1R_1)'$.

Fig. (6) has been drawn for the case when B lies on P_1P produced. In the alternative when B lies on PP_1 produced we have only to interchange the positive and the negative signs in the above proof.

Let $L_2\Omega$ be as before parallel to PL_1. Draw $R_3\Omega$ and $R_4\Omega$ perpendicular to K_1R_1 and K_2R_2 at R_3 and R_4 respectively [See Fig. (7)].

$$K_1R_3 = (BK_1)' = K_2R_2,$$

$$K_1R_1 = (BK_2)' = K_2R_4,$$

* *Cf.* Carslaw, 'Non-Euclidean Geometry', p. 50, § 26.6.

Hence, $R_1R_3 = R_2R_4$.

Supposing the lines $R_1\Omega$, $R_2\Omega$ (not shown in the figure) drawn and noting that in the infinite triangles $L_1R_1\Omega$ and $L_2R_2\Omega$, we have $\angle L_1R_1\Omega = \angle L_2R_2\Omega$ and $\angle R_1L_1\Omega = \angle R_2L_2\Omega$, it follows that

$$L_1R_1 = L_2R_2.$$

Proof of(c). Draw the lines $C\Omega_1$, $C\Omega_2$, $C\Omega_3$.

Now, $\angle L_1C\Omega_2 = \angle L_1C\Omega_1$,

or, $\angle L_1CS_2 + \angle \Omega_2CS_2 = \angle \Omega_1CS_1 + \angle L_1CS_1$,

or, $\angle L_1CS_2 + \pi(CS_2) = \pi(CS_1) + \angle L_1CS_1$.

Again, $\angle L_1CS_2 - \angle \Omega_3CS_2 = \angle \Omega_3CS_1 - \angle L_1CS_1$,

or, $\angle L_1CS_2 - \pi(CS_2) = \pi(CS_1) - \angle L_1CS_1$.

Hence, $\angle L_1CS_1 = \pi(CS_2)$ and $\angle L_1CS_2 = \pi(CS_1)$.

Fig. (8) has been drawn for the case when C lies to the same side of L_1 as P. When C lies on the opposite side we have only to interchange the positive and the negative signs in the above proof.

Let $L_{12}\Omega$ be as before and draw $S_3\Omega$ and $S_4\Omega$ perpendicular to CS_1 and CS_2 at S_3 and S_4 respectively [Fig. (8)]. It is easily seen that $S_1S_3 = S_2S_4$. Reasoning as in part (b) it follows that

$$L_1S_1 = L_2S_2.$$

2. THEOREM II. *From a point* P *on the side* $\Omega_1\Omega_2$, *of a limiting triangle* $\Omega_1\Omega_2\Omega_3$ *perpendiculars* PM_1 *and* PM_2 *are drawn to the sides* $\Omega_1\Omega_3$ *and* $\Omega_2\Omega_3$ *respectively. If a line* s *through* P *meets* $\Omega_1\Omega_3$ *at* T_1 *and possesses a common perpendicular* HT_2 *with* $\Omega_2\Omega_3$ [See Fig. (9)] *then,*

$$\Delta(\angle PT_1M_1) = PH, \quad PT_1 = (HT_2)' \textit{ and } M_1T_1 = M_2T_2.$$

It is seen at once that $\angle M_1PM_2 = \frac{\pi}{2}$. M_1P is therefore the common perpendicular to $\Omega_1\Omega_2$ and PM_2, and M_2P is the common perpendicular to $\Omega_2\Omega_3$ and PM_1.

Draw the lines $E_1\Omega_1$, $E_2\Omega_2$, $E_3\Omega_3$ perpendicular to s at E_1, E_2, E_3 respectively [See Fig. (10)].

Now, $\quad E_1P = PE_2$,

or, $\quad E_1T_1 - PT_1 = PH - E_2H$,

or, $\quad \Delta(\angle PT_1M_1) - PT_1 = PH - (HT_2)'$.

Again, $\quad T_1E_3 + PT_1 = PH + HE_3$,

or, $\quad \Delta(\angle PT_1M_1) + PT_1 = PH + (HT_2)'$,

Hence, $\quad \Delta(\angle PT_1M_1) = PH$ and $PT_1 = (HT_2)'$.

Draw $M\Sigma_3$ perpendicular to PM_2 at M and parallel to PT_1 [See Fig. (11)]. Now $\Sigma_1\Sigma_2$ the common parallel to T_1P and Σ_3M is readily seen to be perpendicular to M_1P. It follows by applying Th. 1 (a) to the limiting triangle $\Sigma_1\Sigma_2\Sigma_3$ that M_1T_1 produced meets $\Sigma_2\Sigma_3$ at a point T such that

$$M_1T_1 = \Delta(\angle MTM_1) \text{ and } PT_1 = MT.$$

Draw $N_2\Sigma_3$ perpendicular to $\Omega_2\Omega_3$ at N_2 and $N\Omega_3$ perpendicular to $\Sigma_2\Sigma_3$ at N [Fig. (11)].

Now, $\quad MN = M_2N_2$,

or, $\quad MT + TN = T_2N_2 + M_2T_2$,

or, $\quad PT_1 + \Delta(\angle MTM_1) = (HT_2)' + M_2T_2$,

or, $\quad PT_1 + M_1T_1 = PT_1 + M_2T_2$.

Hence, $\quad M_1T_1 = M_2T_2$.

The Figs. (10) and (11) have been drawn for the case when s lies in the angular region $M_1P\Omega_3$. In this case H lies on PT_1 or PT_1 produced towards T_1. The same proof with suitable modifications of signs holds in the alternative when s lies in the angular region $M_1P\Omega_1$ and in consequence H lies on T_1P produced towards P.

§ 2. *Correspondences between right-angled triangles tri-rectangular quadrilaterals and rectangular pentagons.*

We shall make it a convention to write down the elements of a right-angled triangle in the following order :—

Hypotenuse, one acute angle, the side adjacent to it, the other side, the other acute angle.

For example the right-angled triangle (T_1) shown in Fig. (12) shall be described as having the elements

$$c, \quad \lambda, \quad b, \quad a, \quad \mu.$$

Similarly the elements of a tri-rectangular quadrilateral will be written in the following order :—

One side adjacent to the acute angle, the acute angle, the other side adjecent to it, the side opposite to the one first written, the remaining side.

For example the tri-rectangular quadrilateral (Q_1) shown in Fig. (12) shall be described as having the elements

$$c, \quad \beta, \quad l, \quad a, \quad m'.$$

A rectangular pentagon shall be described by writing down its sides in order. For example the rectangular pentagon (P) shown in Fig. (12) may be said to have the elements

$$c, \quad b', \quad l, \quad m, \quad a'.$$

*1. Lobatschewsky's correspondence between a right angled triangle and a tri-rectangular quadrilateral.**

THEOREM I. *To any right-angled triangle*(T_1) [Fig. (12)] *with elements*

$$c, \quad \lambda, \quad b, \quad a, \quad \mu,$$

there corresponds a tri-rectangular quadrilateral(Q_1) [Fig. (12)] *with elements*

$$c, \quad \beta, \quad l, \quad a, \quad m'.$$

Identify (T_1) with the right-angled triangle AQ_1L_1 of Fig. (1) by setting

$$L_1Q_1=c, \quad \angle L_1Q_1A=\lambda, \quad Q_1A=b, \quad AL_1=a, \quad \angle AL_1Q_1=\mu.$$

Complete the figure by drawing $P\Omega_1$ perpendicular to AL_1 at P and parallel to Q_1L_1, and $\Omega_2\Omega_3$ the common parallel to Ω_1P and L_1Q_1. If AQ_1 meets $\Omega_2\Omega_3$ at Q_2 and L_2 is the foot of the perpendicular from L_1 to $\Omega_2\Omega_3$, we have from §1. Th. I (a),

$$AQ_2=l, \quad \angle AQ_2L_2=\beta \text{ and } Q_2L_2=c.$$

* Nikolai Jwanwitsch Lobaschfeskij, 'Zwei Geometrische etc.', *Loc. Cit.*, p. 25 and p. 256.

Also $\angle AL_1L_2=\frac{\pi}{2}$. Hence $\angle \Omega_3L_1L_2=\frac{\pi}{2}-\mu$ and $L_1L_2=m$.

Thus $L_2Q_2\,AL_1$ is the tri-rectangular quadrilateral (Q_1).

Conversely if we start by identifying (Q_1) with the quadrilateral $L_2Q_2AL_1$ of Fig. (1), we can by completing the figure, derive the associated right-angled triangle (T_1).

*2. Mukhopadhyaya's correspondence between a tri-rectangular quadrilateral and a rectangular pentagon.**

THEOREM II. *To any tri-rectangular quadrilateral* (Q_1) [Fig. (12)] *with elements*

$$c, \quad \beta, \quad l, \quad a, \quad m',$$

there corresponds a rectangular pentagon (P) [Fig. (12)] *with elements*

$$c, \quad b', \quad l, \quad m, \quad a'.$$

Identify (Q_1) with the tri-rectangular quadrilateral $BL_1R_1K_1$ of Fig. (2) by setting

$$BL_1=c, \quad \angle BL_1R_1=\beta, \quad L_1R_1=l, \quad R_1K_1=a, \quad K_1B=m'.$$

Complete the figure by drawing $P\Omega_1$ perpendicular to BL_1 at P and parallel to L_1R_1, and $\Omega_2\Omega_3$ the common parallel to Ω_1P and R_1L_1. If BK_1 and $\Omega_2\Omega_3$ possess the common perpendicular K_2R_2, and L_2 be the foot of the perpendicular from L_1 to $\Omega_2\Omega_3$, we have §1. Th I (*b*)

$$K_2R_2=m, \; BK_2=d \text{ and } L_2R_2=c.$$

Also $\angle BL_1L_2=\frac{\pi}{2}$. Hence $\angle \Omega_3L_1L_2=\frac{\pi}{2}-\beta$ and $L_1L_2=b'$.

Thus $BL_1L_2R_2K_2$ is the rectangular pentagon (P).

Conversely, starting with (P) we can derive (Q_1).

* S. Mukhopadhyaya, *Loc. cit*

Corollary. The five right-angled triangles (T_1), (T_2), (T_3), (T_4), (T_5), *the five tri-rectangular quadrilaterals* (Q_1), (Q_2), (Q_3), (Q_4), (Q_5) *and the rectangular pentagon* (P) *whose elements are given below, form a system of associated figures.*

(T_1)	c,	λ,	b,	a,	μ.
(T_2)	l,	$\frac{\pi}{2}-a$,	m',	b,	γ.
(T_3)	a',	$\frac{\pi}{2}-\beta$,	c',	m',	λ.
(T_4)	b',	μ,	l',	c',	$\frac{\pi}{2}-a$.
(T_5)	m,	γ,	a,	l',	$\frac{\pi}{2}-\beta$.
(Q_1)	c,	β,	l,	a,	m'.
(Q_2)	l,	$\frac{\pi}{2}-\mu$,	a',	b,	c'.
(Q_3)	a',	$\frac{\pi}{2}-\gamma$,	b',	m',	l'.
(Q_4)	b',	$\frac{\pi}{2}-\lambda$,	m,	c',	a.
(Q_5)	m,	a,	c,	l',	b.
(P)	c,	b',	l,	m,	a'.

We have already established that (T_1), (Q_1) and (P) are associated figures. Now the elements of (P) can be written in five different ways, *viz.*

(1)	c,	b',	l,	m,	a'.
(2)	l,	m,	a',	c,	b'.
(3)	a',	c,	b',	l,	m.
(4)	b',	l,	m,	a',	c.
(5)	m,	a',	c,	b',	l.

If (T_1) be regarded as corresponding to the first of these ways of writing the elements of (P), then to each of the other four ways there correspond in succession the right-angled triangles (T_2), (T_3), (T_4) and (T_5). Similarly if (Q_1) be regarded as corresponding to the first way, then to the other four ways there correspond the tri-rectangular quadrilaterals (Q_2), (Q_3), (Q_4), and (Q_5).

§ 3. *A transformation carrying any Triangle into an associated Triangle.*

In order to introduce generality into our statements we have to introduce besides proper points, improper points and proper lines, ideal points, improper lines and ideal lines.*

All lines perpendicular to a given line p, must be regarded as passing through a common *ideal point* represented by the line p. An ideal point will be denoted by a Greek capital letter with a suffix to denote the line to which the ideal point corresponds *e.g.*, π_p and Σ_s denote ideal points represented by the lines p and s respectively.

If Ω is an improper point, then every line of the pencil of parallels through Ω is the representative entity of an ideal point. The locus of these ideal points, together with the point Ω is an *improper line—line at infinity*. It will be denoted by ω.

If B is a proper point every line through B is the representative entity of an ideal point. The locus of these ideal points is an *ideal line*. It will be denoted by β_B. The point B can be taken as the representative of the ideal line β_B.

We can now state,

(*i*) Through any two points passes one and only one line.

(*ii*) Any two lines pass through one and only one common point.

Any three points, (proper, improper or ideal) may be regarded as the vertices of a *Triangle* (we shall always begin the word *Triangle* with T when using it in this *general* sense). The three lines joining the points in pairs are then the sides of this Triangle. Conversely any three lines (proper, improper or ideal) may be regarded as the sides of a Triangle and their points of intersection as its vertices.

From the general point of view adopted here, right-angled triangles, tri-rectangular quadrilaterals, and rectangular pentagons are to be regarded as Right-angled Triangles. We have in § 2 investigated the correspondences between Right-angled Triangles. The transformation explained below helps us to generalise these results to any Triangles whatsoever.

* *Cf.* Bonola, 'La Geometria non-Euclidea' (*English Translation*) by Carslaw, Appendix IV, p. 227.

1. *To transform a Triangle into another associated to it.*

Let (T) be a Triangle having the proper line u for one of its sides. The other two sides may be proper, improper or ideal. On u take any two proper points Q and R so that QR and RQ are the two rays defined by u. Through the vertex of (T) opposite to u draw the line w parallel to the ray QR. Let v be the third side of the limiting triangle two of whose sides are u and w. Then *the Triangle* (T′), obtained by replacing the side u of (T) by v and retaining the other two sides, *will be called the transform of* (T) *with respect to the ray* QR. (*i*) Let the vertex opposite to u be a proper point P. The elements of (T′) are then determined in terms of the corresponding elements of (T) by § 1. Th. II. (*ii*) Let the vertex opposite to u be an ideal point π, represented by the line p. The lines u, v, w form the sides of a limiting triangle, and p is perpendicular to w. Hence p either meets one of u and v, or else is the middle parallel between them. In the former case the elements of (T′) are determined in terms of the corresponding elements of (T) by § 1. Th. I. In the latter case (T) and (T′) are congruent and their corresponding elements are equal. (*iii*) The transformation becomes illusory when the vertex opposite to u is improper.

Thus any Triangle can be transformed with respect to one of the two rays defined by a proper side (the opposite vertex not being improper). The transformed Triangle is then completely determined as regards the position of its sides and vertices and each one of its elements is unambiguously fixed by a corresponding element of the original Triangle.

Notice that if (T″) is the transform of (T) with respect to the ray RQ, then (T″) is congruent to (T).

2. *The reverse transformation.*

It is easy to find the transformation reverse to that explained above. If Ω_1 be the improper point common to w and v, and Ω_2 the improper point common to u and v, then (T′) transformed with respect to the ray $\Omega_2\Omega_1$ goes over into the original Triangle (T).

§ 4. *Associated Triangles.*

According to the point of view adopted in § 3, we must include triangles, bi-rectangular quadrilaterals, tetra-rectangular pentagons and rectangular hexagons, simple or crossed, under the common name of Triangles.

A bi-rectangular quadrilateral shall be said to be of the first or the second type according as the two right angles are or are not adjacent to

the same side, *e.g.*, the bi-rectangular quadrilateral (Q) of Fig. (22) is of the first type, while the bi-rectangular quadrilateral (U) of Fig. (20) is of the second type.

It should be specially noted that in the theorems I-VI that follow, λ, μ, ν, α, β, γ, δ indicate angles which do not exceed a right angle and consequently their supplements indicate angles which are not less than right angles.

1. THEOREM I. *The triangle* (A) *and the tetra-rectangular pentagons* (B), (C), (D) *whose elements are shown in* Fig. (18) *from a system of associated figures.*

Let us transform the triangle (A) as explained in § 3.1, with respect to the ray Q_1R_1. Draw $P_1\Omega_1$ parallel to Q_1R_1 [See Fig. (14)] and $\Omega_2\Omega_3$ the common parallel to Ω_1P_1 and R_1Q_1. Then the transformed Triangle has for its sides the lines P_1Q_1, P_1R_1 and $\Omega_2\Omega_3$. Let P_1M_1, P_1M_2 be the perpendiculars drawn from P_1 to $\Omega_1\Omega_3$ and $\Omega_2\Omega_3$ respectively. Then $\angle M_1P_1M_2=\frac{\pi}{2}$. Also since μ and ν are acute, P_1M_1 lies in the angular region $Q_1P_1R_1$. P_1Q_1 therefore makes an acute angle with P_1M_2. Consequently if UX be the common perpendicular to P_1Q_1 and $\Omega_2\Omega_3$, then U lies on P_1Q_1 or the prolongation of P_1Q_1 towards Q_1. Similarly we can prove that if VY is the common perpendicular to R_1P_1 and $\Omega_3\Omega_2$, V lies on the prolongation of R_1P_1 towards P_1. The trnsformed Triangle is then none other than the tetra-rectangular pentagon P_1UXYV. From §1. Th. II we have

$$P_1U=m,\qquad UX=c',\qquad YV=b',\qquad VP_1=n.$$

Also $XY=XM_2+M_2Y=Q_1M_1+M_1R_1=c$, and $\angle UP_1V=\pi-\lambda$.

Thus the pentagon P_1UXYV is congruent to the pentagon (B) of Fig. (13).

Similarly transforming (A) with respect to the rays R_1P_1 and P_1Q_1 respectively, we obtain figures congruent to (C) and (D).

Corollary (1). In the limiting case when $\nu=\frac{\pi}{2}$, the triangle (A) becomes the right-angled triangle (T_1) of §2, (D) becomes the associated rectangular pentagon (P), while (B) and (C) reduce to the tri-rectangular quadrilaterals (Q_3) and (Q_4) respectively. The correspondence between figures (T_1), (P), (Q_3) and (Q_4) of §2 is thus a particular case of the correspondence established here.

Corollary (2). If we remember that two parallel straight lines may be regarded as including a null angle, and possessing a vanishing common perpendicular at infinity, then by making in the above theorem the angle ν null, and at the same time segments a and b infinite, (A) reduces to an infinite triangle, and (B) and (C) to two infinite bi-rectangular quadrilaterals of the first type, whose elements are shown in Fig. (15) The tetra-rectangular pentagon (D) becomes illusory.

The correspondences between the infinite triangle (A), and the infinite bi-rectangular quadrilaterals (B) and (C) of Fig. (15) can however be established directly. In fact §1. Th. II. shows as before that the transforms of (A) [Fig. (15] with respect to the rays $Q_1\Sigma_1$ and $P_1\Sigma_1$ are respectively congruent to (B) and (C) [Fig. (15)].

THEOREM II. *The triangle* (E) *and the tetra-rectangular pentagons* (F), (G), (H) *whose elements are shown in* Fig. (16) *from a system of associated figures.*

The proof is similar to that of Theorem I.

Corollary. If we make ν null and at the same time the segments b and c infinite, we get a correspondence between an infinite triangle (with an obtuse angle), and two infinite bi-rectangular quadrilaterals of the first type, one simple and the other crossed. It is easy to write down the elements of these figures. The pentagon (H) becomes illusory.

2. THEOREM III. *The rectangular hexagons* (K), (L), (M), (N) *whose elements are shown in* Fig. (17), *form a system of associated figures.*

The hexagon (K) may be regarded as a Triangle with proper sides B_1C_1, D_1E_1 and F_1A_1. Let us transform it with respect to the ray D_1E_1. To do this we draw $P\Omega_3$ perpendicular to A_1B_1 at P and parallel to D_1E_1, and $\Omega_1\Omega_3$ the common parallel to Ω_3P and E_1D_1, [See Fig. (18)]. The transformed Triangle then has for its sides the lines B_1C_1, F_1A_1 and $\Omega_1\Omega_3$. Let UV be the common perpendicular to the ultra parallel lines B_1C_1 and $\Omega_1\Omega_3$, and XY the common perpendicular to ultra parallel lines $\Omega_1\Omega_3$ and F_1A_1. The transform is then the rectangular hexagon A_1B_1UXVY. An application of §1. Th. I. b) now shows that

$$B_1U=b', \qquad UV=n', \qquad VX=l, \qquad XY=m', \qquad YA_1=c'.$$

Thus the hexagon A_1B_1UVXY is congruent to the rectangular hexagon (L) of Fig. (17).

Similarly transforming (K) with respect to the rays F_1A_1 and B_1C_1 respectively we obtain figures congruent to (M) and (N).

Corollary. If the segments a and b are made infinite and at the same time the segment n is made to vanish, we get a correspndence between a simple infinite tetra-rectangular pentagon, and two crossed infinite tetra-rectangular pentagons. The hexagon (N) becomes illusory.

3. THEOREM IV. *The three bi-rectangular quadrilaterals* (U), (V) *and* (W), *of the second type, whose elements are shown in* Fig. (20) *form a system of associated figures.*

The quadrilateral (U) may be regarded as a Triangle with proper sides A_1D_1, C_1D_1 and an ideal side β_{B_1} represented by the point B_1. The vertex opposite the side A_1D_1 is ideal being represented by the line B_1C_1. Transform (U) with respect to the ray D_1A_1. To do this draw $P\Omega_1$ perpendicular to B_1C_1 at P and parallel to D_1A_1, and $\Omega_2\Omega_3$ the common parallel to Ω_1P and A_1D_1 [See Fig. (19)]. Let Y be the foot of the perpendicular from B_1 to $\Omega_2\Omega_3$ and D_1C_1 produced meet $\Omega_2\Omega_3$ at X. The transformed Triangle has for its sides β_{B_1}, C_1D_1 and $\Omega_2\Omega_3$ *i.e.*, it is the bi-rectangular quadrilateral C_1XYB.

In case $n=d$, B_1C_1 is parallel to A_1D_1 and $l=b$, the quadrilaterals (U) and (V) are congruent. It is easy to see that in Fig. (19) B_1C_1 is the middle parallel between $\Omega_1\Omega_3$ and $\Omega_2\Omega_3$. Hence B_1C_1XY is congruent to $B_1C_1D_1A_1$. It follows that the transformed bi-rectangular quadrilateral B_1C_1XY is congruent to (V).

The same result follows when $n \neq d$. We note that in this case B_1C_1 meets $\Omega_1\Omega_2$ or $\Omega_2\Omega_3$ according as $n < d$ or $n > d$ From the point of intersection we now drop a perpendicular to the other and apply §1. Th. I (a) and (c).

Similarly transforming (U) with respect to the ray D_1C_1 we obtain a figure congruent to (W).

Corollary (1). If we put $\delta=\frac{\pi}{2}$ and $\nu=\frac{\pi}{2}-\mu$, (U), (V), (W) respectively become the figures (Q_1), (T_1) and (T_2) of §2.

Corollary (2). If we make the angle β null and at the same time the segments l and c infinite, we get a correspondence between three infinite bi-rectangular quadrilaterals of the first type (having an obtuse angle).

THEOREM V. *The three bi-rectangular quadrilaterals* (X), (Y), (Z), *of the second type, whose elements are shown in* Fig. (21) *form a system of associated figures.*

The proof is similar to that of theorem IV.

Corollary. Making the angle β null and at the same time the segments l and c infinite, we get a correspondence between three infinite bi-rectangular quadrilaterals of the second type (having an acute angle), one of which is simple and the other two crossed.

THEOREM VI. *The four bi-rectangular quadrilaterals* (Q), (R), (S), (T), *of the first type, whose elements are shown in* Fig. (22), *form a system of associated figures.*

The quadrilateral (Q) may be regarded as a Triangle with proper sides A_1B_1, B_1C_1, C_1D_1 Transforming it with respect to the rays A_1B_1, D_1C_1 and B_1C_1 respectively, we obtain figures congruent to (R), (S) and (T).

Corollary. If we make $\delta = \frac{\pi}{2}$ and $\nu = \frac{\pi}{2} - \mu$, (Q), (R), (S), (T) respectively become the figures (Q_1), (T_3), (Q_2) and (T_1) of §2.

§ 5. *To construct a Triangle with three angles or three sides given.*

In order to understand the scope of the problem the existence of various kinds of distances and angles must be noted. For example the angle between a proper line p and an ideal line γ_e will be measured by the perpendicular from the point C to the line p. Since the quadrilateral (U) of Fig. (20) may be regarded as a Triangle with sides A_1D_1, C_1D_1 and β_{B_1}, the problem of constructing (U), when c, l and δ are given may be regarded as a particular case of the general problem of constructing a Triangle with three given angles.

1. *Remarks on the assumption of continuity.*

The fundamental problem of Hyperbolic Geometry is to find the angle $\pi(c)$ having given the segment c. In order to solve this problem, the possibility of drawing a circle with a given radius has to be assumed though we do not require any hypothesis as to the intersection of straight lines and circles, *i.e.*, we have no need to invoke the Principle of Continuity.* The converse problem, *viz.*, that of finding the segment $\Delta(\gamma)$ when the angle γ is given can be made to depend on the fundamental problem†. In the constructions that follow we shall have

* *Cf.* Carslaw, *Loc. Cit.*, §48, p. 74.

† *Cf.* Hilbert, 'Grundlagen der Geometrie' sechste Auflage Anhang III, p. 161, and Nikolai Jwanwitsch Lobaschfeskij, 'Zwei Geometrische etc,' *Loc. Cit.*, p. 242.

no need of drawing a circle except when as subsidiary to the problem in hand we have to find the angle of parallelism corresponding to a given segment or *vice versa*. *Our constructions therefore do not depend on the Principle of Continuity.*

2. *To construct a Triangle with three given angles (or sides).*

Let (T) be a Triangle for which the three angles (or the three sides) are given. Transform it as explained in §3.1, with respect to one of the two rays defined by a proper side, the vertex opposite which is not improper. It will be found that in the transformed Triangle (T′) two sides and the included angle (or two angles and the included side) are known. We can therefore at once construct (T′). Retransforming (T′) [*vide*, §3.2] we now obtain the required Triangle (T).

For example let it be required to construct the triangle $P_1Q_1R_1$ of Fig (13), having given the angles λ, u, v. Transforming the triangle with respect to the ray Q_1R_1, we obtain as in §4.1 Th. I a tetra-rectangular pentagon P_1UXYV [Fig. (14)]. For this pentagon the angle $UP_1V(=\pi-\lambda)$, and the bounding sides $UP_1(=m)$ and VP_1 $(=n)$ are known. Constructing this pentagon and transforming it with respect to XY we now obtain the required triangle.

The same transformations suffice if instead of the angles λ, μ, v we know the sides a, b, c of $P_1Q_1R_1$ [Fig. (13)]. The only difference is that in the transformed pentagon P_1UXYV, instead of the angle UP_1V and its bounding sides, we now know the side XY, $(=a)$, and the segments $XU(=c')$ $YV(=b')$ defining the adjacent ideal angles, when P_1UXYV is regarded as a Triangle.

Similarly the problem of constructing the rectangular hexagon $A_1B_1C_1D_1E_1F_1$ of Fig. (17), having given the sides a, b, c can be solved by first transforming it with respect to the ray D_1E. We now obtain as in §4.2 Th. III, a crossed rectangular hexagon A_1B_1UVXY [Fig. (18)] for which $A_1B_1(=a)$, $B_1U(=c')$ and A_1Y $(=b')$ are known. Constructing this hexagon and transforming it with respect to the ray VX we obtain the required hexagon.

Again to construct the quadrilateral $A_1B_1C_1D_1$ of Fig. (20) having given c, l and δ we first transform it into an associated quadrilateral [*vide* § 4.3 Th. IV] which can be constructed and retransformed to the required quadrilateral.

It is needless to multiply examples since the same method is found to be effective in every case.

Bull. Cal. Math. Soc., Vol. XIX, No. 3 (1928).

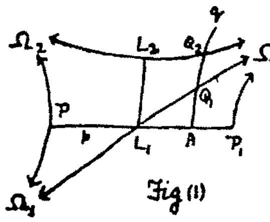


Fig (1)

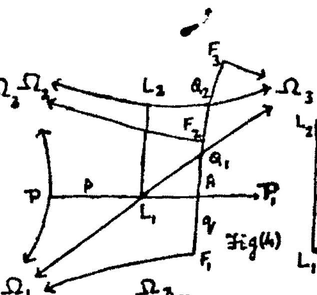


Fig (4)

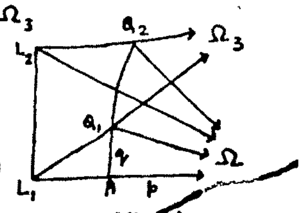
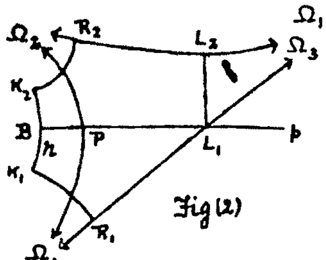


Fig (5)



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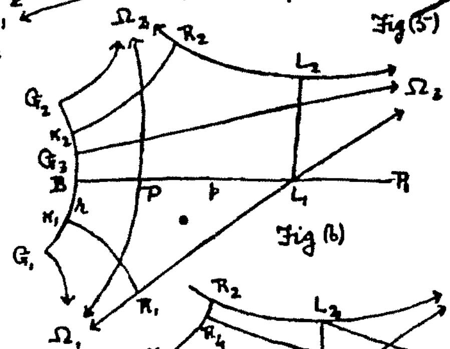
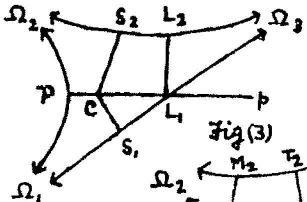


Fig (6)



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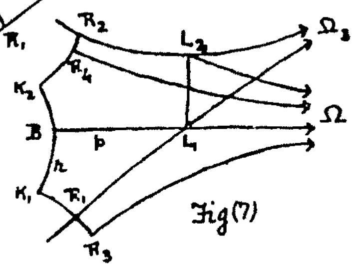

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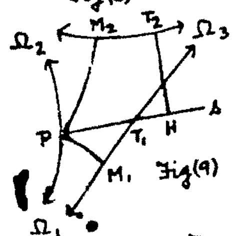


Fig (9)

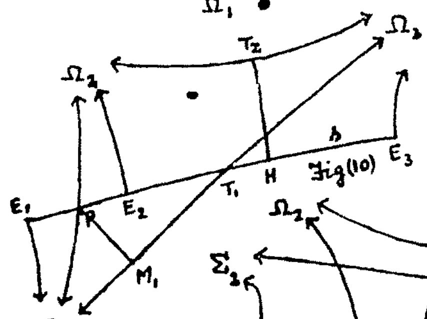


Fig (10)

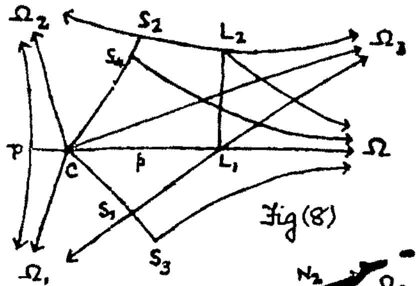


Fig (8)

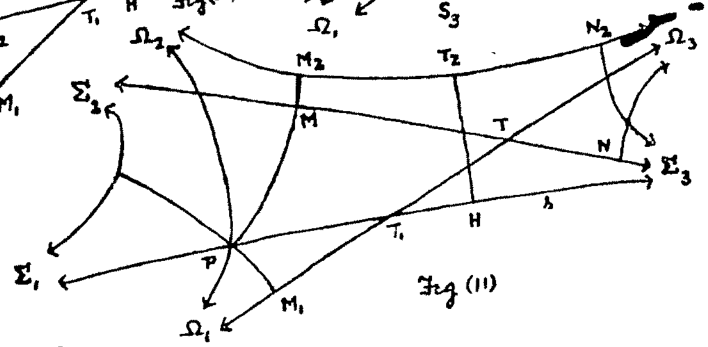


Fig (11)

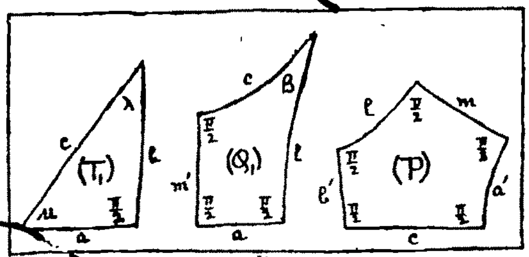


Fig.(12)

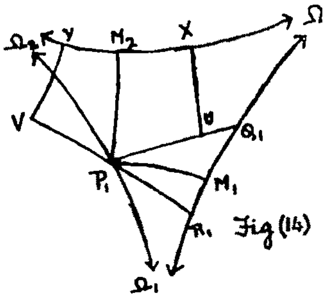


Fig (14)

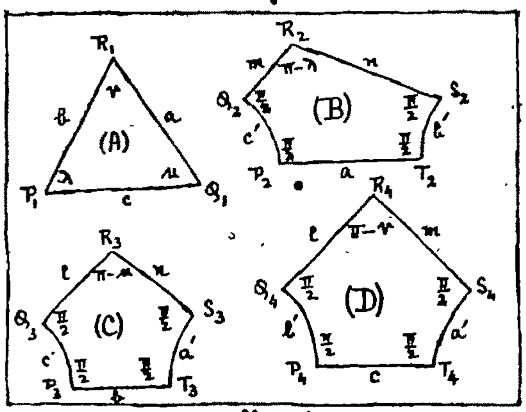


Fig (13)

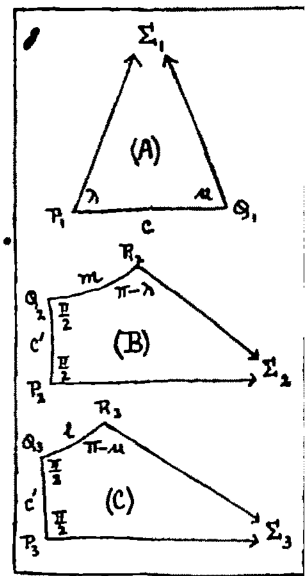
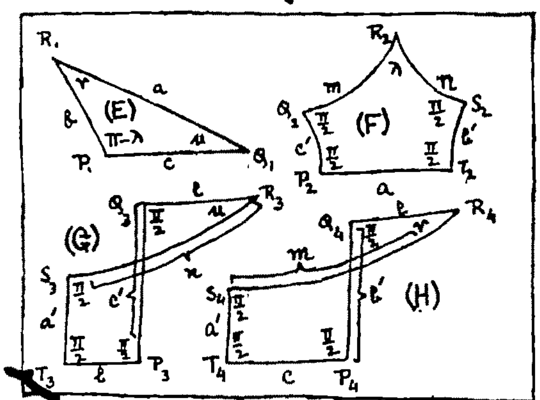


Fig.(15)



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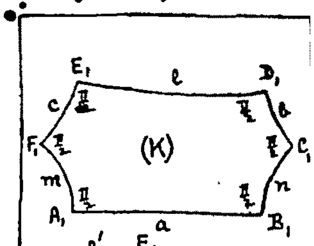

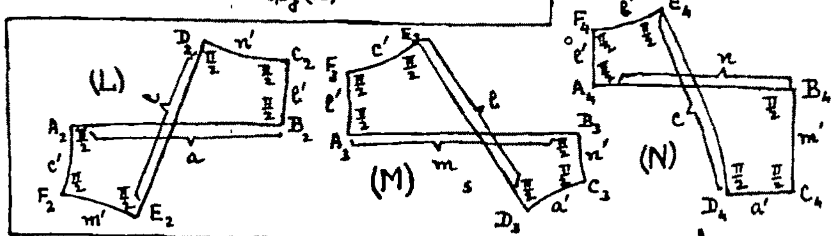

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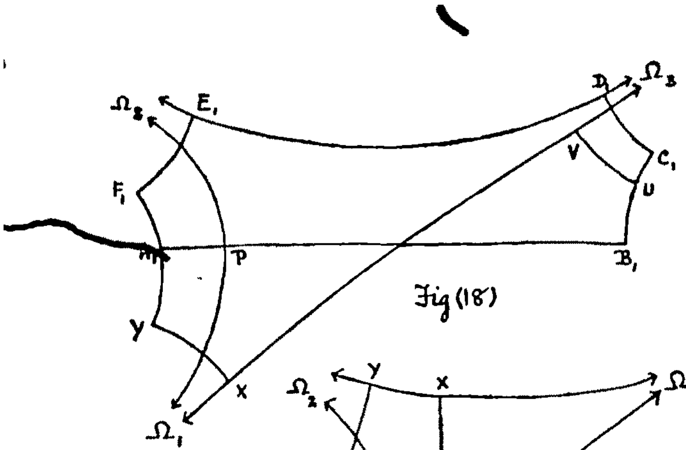

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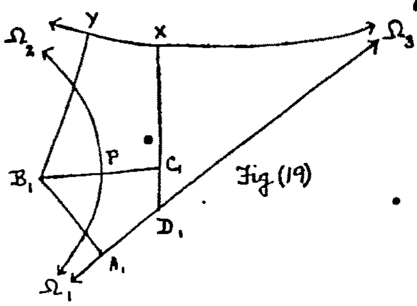

Fig (19)

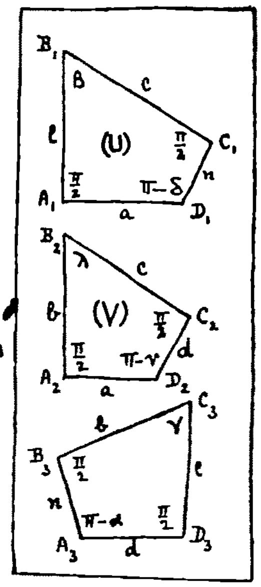

Fig (20)

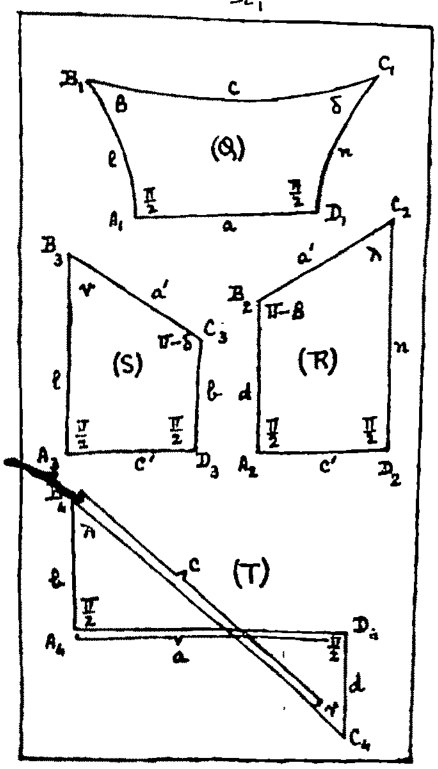

Fig (22)

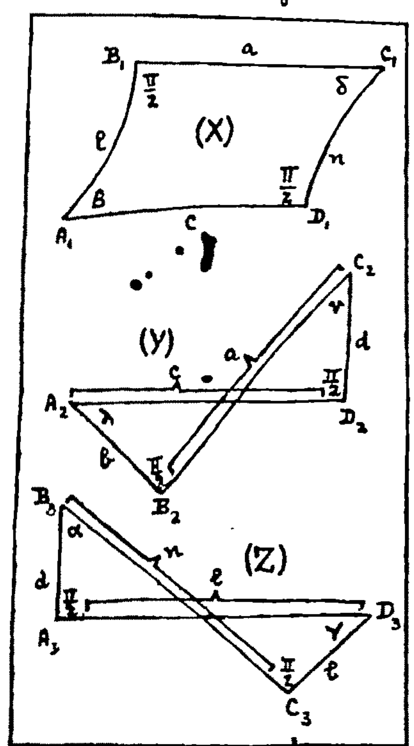

Fig (21)

2

ON PLANE STRAIN AND STRESS IN ROTATING ELLIPTIC CYLINDERS AND DISKS

BY

SUDDHODAN GHOSH

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The problem of the determination of stress and strain in rotating elliptic cylinders and disks has been worked out by Chree* as an application of his general solution† of the equations of Elasticity. But, as he uses cartesian co-ordinates, his method involves a large amount of calculations and his expressions are often far from being simple. The natural co-ordinates for this problem are, however elliptic co-ordinates and solutions in elegant forms can be obtained in approximate agreement with the boundary conditions in these co-ordinates. This problem has also been attempted in a recent paper‡ by Dr. N. M. Basu and Mr. H. M. Sen Gupta, but their solution does not satisfy the maximum number of boundary conditions and is only applicable to slowly rotating cylinders with nearly circular sections. In the present paper no such limitation is made and the problem has been considered as a case of plane strain (for cylinder) or plane stress (for disk) and the stress function χ introduced with advantage. In the case of rotating cylinders this function χ satisfies an equation of the form $\nabla_1^4 \chi$=constant.

* *Phil. Mag.*, Vol. 34, 1892.

† *Quarterly Journal*, Vol. XXII, p. 89 ; Vol. XXIII, p. 11.

‡ *Bull. Cal. Math. Soc.*, Vol. XVIII, p. 141.

I.

1. If a cylinder rotating about an axis parallel to a generator with angular velocity ω be in a state of plane strain, its equations of motion can be written in terms of dilatation and rotation as follows :

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left[(\lambda+2\mu)\Delta+\tfrac{1}{2}\rho\omega^2 r^2\right]-2\mu\frac{\partial \varpi}{\partial y}=0 \\ \frac{\partial}{\partial y}\left[(\lambda+2\mu)\Delta+\tfrac{1}{2}\rho\omega^2 r^2\right]+2\mu\frac{\partial \varpi}{\partial x}=0 \end{aligned}\right\} \quad \ldots \quad (1)$$

where $r^2=x^2+y^2$. From (1) it is evident that

$$(\lambda+2\mu)\Delta+\tfrac{1}{2}\rho\omega^2 r^2 \text{ and } 2\mu\varpi$$

are conjugate harmonic functions.

The stress equations of motion are

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left[X_x+\tfrac{1}{2}\rho\omega^2 r^2\right]+\frac{\partial X_y}{\partial y}=0 \\ \frac{\partial X_y}{\partial x}+\frac{\partial}{\partial y}\left[Y_y+\tfrac{1}{2}\rho\omega^2 r^2\right]=0 \end{aligned}\right\} \quad \ldots \quad (2)$$

showing that the stresses are derivable from a stress function χ as follows :

$$X_x+\tfrac{1}{2}\rho\omega^2 r^2=\frac{\partial^2\chi}{\partial y^2},\ Y_y+\tfrac{1}{2}\rho\omega^2 r^2=\frac{\partial^2\chi}{\partial x^2},\ X_y=-\frac{\partial^2\chi}{\partial x\,\partial y}. \quad \ldots \quad (3)$$

These give

$$X_x+Y_y+\rho\omega^2 r^2=\nabla_1^2\chi$$

and since $$X_x+Y_y=2(\lambda+\mu)\Delta-2\mu e$$

where e is the extension along the generators of the cylinder

$$2(\lambda+\mu)\Delta=\nabla_1^2\chi-\rho\omega^2r^2+2\mu e. \qquad \dots \quad (4)$$

Now $(\lambda+2\mu)\Delta+\frac{1}{2}\rho\omega^2r^2$ being a plane harmonic

$\nabla_1^2\chi-\frac{\mu}{\lambda+2\mu}\rho\omega^2r^2+2\mu e$ is also a plane harmonic

so that χ satisfies the equation

$$\nabla_1^4\chi=\frac{4\mu}{\lambda+2\mu}\rho\omega^2. \qquad \dots \quad (5)$$

If we use two-dimensional elliptic co-ordinates given by

$$x=c\cosh\alpha\cos\beta,\quad y=c\sinh\alpha\sin\beta$$

so that

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2=\frac{c^2}{2}(\cosh 2\alpha-\cos 2\beta) \qquad \dots \quad (6)$$

a particular solution of (5) can be found as

$$\chi=\frac{\mu\rho\omega^2c^4}{128(\lambda+2\mu)}(\cosh 4\alpha+\cos 4\beta). \qquad \dots \quad (7)$$

Hence we can write

$$\chi=\chi_0+\frac{\mu\rho\omega^2c^4}{128(\lambda+2\mu)}(\cosh 4\alpha+\cos 4\beta) \qquad \dots \quad (8)$$

where

$$\nabla_1^4 \chi_0 = 0. \qquad \dots \quad (9)$$

2. We have the stress system in curvilinear co-ordinates

$$\left.\begin{aligned}
\widehat{\alpha\alpha} &= h\frac{\partial}{\partial\beta}\left(h\frac{\partial\chi}{\partial\beta}\right) - h\frac{\partial h}{\partial\alpha}\frac{\partial\chi}{\partial\alpha} - \tfrac{1}{2}\rho\omega^2 r^2 \\
\widehat{\beta\beta} &= h\frac{\partial}{\partial\alpha}\left(h\frac{\partial\chi}{\partial\alpha}\right) - h\frac{\partial h}{\partial\beta}\frac{\partial\chi}{\partial\beta} - \tfrac{1}{2}\rho\omega^2 r^2 \\
\widehat{\alpha\beta} &= -h^2\frac{\partial^2\chi}{\partial\alpha\partial\beta} - h\frac{\partial h}{\partial\alpha}\frac{\partial\chi}{\partial\beta} - h\frac{\partial h}{\partial\beta}\frac{\partial\chi}{\partial\alpha} \\
\widehat{zz} &= \frac{\lambda}{2(\lambda+2\mu)}\left[\nabla_1^2\chi - \rho\omega^2 r^2\right] + 2\mu e
\end{aligned}\right\} \qquad \dots \quad (10)$$

Also in terms of the strain components

$$\left.\begin{aligned}
\widehat{\alpha\alpha} &= \lambda(e_{\alpha\alpha}+e_{\beta\beta}) + 2\mu e_{\alpha\alpha} + \lambda e \\
\widehat{\beta\beta} &= \lambda(e_{\alpha\alpha}+e_{\beta\beta}) + 2\mu e_{\beta\beta} + \lambda e \\
\widehat{\alpha\beta} &= \mu e_{\alpha\beta} \\
\widehat{zz} &= \lambda(e_{\alpha\alpha}+e_{\beta\beta}) + (\lambda+2\mu)e
\end{aligned}\right\} \qquad \dots \quad (11)$$

where

$$\left.\begin{aligned}
&e_{\alpha\alpha} = h\frac{\partial u}{\partial\alpha} - v\frac{\partial h}{\partial\beta}, \quad e_{\beta\beta} = h\frac{\partial v}{\partial\beta} - u\frac{\partial h}{\partial\alpha}, \quad e_{zz} = \frac{\partial w}{\partial z} \\
&e_{\alpha\beta} = \frac{\partial}{\partial\alpha}(hv) + \frac{\partial}{\partial\beta}(hu), \quad e_{\beta z} = 0, \quad e_{z\alpha} = 0
\end{aligned}\right\} \qquad \dots \quad (12)$$

Equating $\widehat{\alpha\alpha}+\widehat{\beta\beta}$, $\widehat{\alpha\alpha}-\widehat{\beta\beta}$ and $\widehat{\alpha\beta}$ from (10), (11) and (12) we have

$$\left.\begin{aligned} & h^2\left(\frac{\partial^2\chi}{\partial\alpha^2}+\frac{\partial^2\chi}{\partial\beta^2}\right)-\rho\omega^2r^2 \\ & \qquad =2(\lambda+\mu)h^2\left[\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)+\frac{\partial}{\partial\beta}\left(\frac{v}{h}\right)\right]+2\lambda e \\ & \frac{\partial}{\partial\alpha}\left[h^2\frac{\partial\chi}{\partial\alpha}+2\mu uh\right]-\frac{\partial}{\partial\beta}\left[h^2\frac{\partial\chi}{\partial\beta}+2\mu vh\right]=0 \\ & \frac{\partial}{\partial\alpha}\left[h^2\frac{\partial\chi}{\partial\beta}+2\mu vh\right]+\frac{\partial}{\partial\beta}\left[h^2\frac{\partial\chi}{\partial\alpha}+2\mu uh\right]=0 \end{aligned}\right\} \quad \ldots \quad (13)$$

The last two equations of (13) can be written as

$$\left.\begin{aligned} 2\mu\frac{u}{h} &= -\frac{\partial\chi}{\partial\alpha}+\frac{1}{h^2}\,\frac{\partial P}{\partial\beta} \\ 2\mu\frac{v}{h} &= -\frac{\partial\chi}{\partial\beta}+\frac{1}{h^2}\,\frac{\partial P}{\partial\alpha} \end{aligned}\right\} \quad \ldots \quad (14)$$

where

$$\nabla_1^2 P=0. \quad \ldots \quad (15)$$

Substituting in the first equation of (13) we obtain for P also the equation

$$\frac{\lambda+2\mu}{\mu}h^2\left(\frac{\partial^2\chi}{\partial\alpha^2}+\frac{\partial^2\chi}{\partial\beta^2}\right)-\rho\omega^2r^2$$

$$=\frac{\lambda+\mu}{\mu}h^2\left\{\frac{\partial}{\partial\alpha}\left(\frac{1}{h^2}\,\frac{\partial P}{\partial\beta}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h^2}\,\frac{\partial P}{\partial\alpha}\right)\right\}+2\lambda e$$

which again reduces to

$$\frac{\lambda+2\mu}{\mu}h^2\left(\frac{\partial^2\chi_0}{\partial\alpha^2}+\frac{\partial^2\chi_0}{\partial\beta^2}\right)$$

$$=\frac{\lambda+\mu}{\mu}h^2\left\{\frac{\partial}{\partial\alpha}\left(\frac{1}{h^2}\,\frac{\partial P}{\partial\beta}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h^2}\,\frac{\partial P}{\partial\alpha}\right)\right\}+2\lambda e. \quad \ldots \quad (16)$$

In terms of elliptic co-ordinates the stresses in (11) can be written as

$$\left.\begin{aligned}\frac{2}{c^2}\frac{\widehat{\alpha\alpha}}{h^4}&=(\cosh 2\alpha-\cos 2\beta)\frac{\partial^2\chi}{\partial\beta^2}+\sinh 2\alpha\frac{\partial\chi}{\partial\alpha}\\&\qquad-\sin 2\beta\frac{\partial\chi}{\partial\beta}\\&\quad-\frac{1}{16}\rho\omega^2c^6(\cosh 2\alpha+\cos 2\beta)(\cosh 2\alpha-\cos 2\beta)^2\\\frac{2}{c^2}\cdot\frac{\widehat{\alpha\beta}}{h^4}&=-(\cosh 2\alpha-\cos 2\beta)\frac{\partial^2\chi}{\partial\alpha\partial\beta}+\sinh 2\alpha\frac{\partial\chi}{\partial\beta}\\&\qquad+\sin 2\beta\frac{\partial\chi}{\partial\alpha}\end{aligned}\right\}\quad\ldots\quad(17)$$

3. Let us apply the above solution to a cylinder with elliptic boundary $\alpha=\alpha_0$. Then at this boundary

$$\widehat{\alpha\alpha}=0,\quad \widehat{\alpha\beta}=0. \qquad\ldots\quad(18)$$

Assume the following solution of (9)

$$\chi_0=a_0\cosh 2\alpha+(\bar{a}_0+a'_2\cosh 2\alpha+a_2\cosh 4\alpha)\cos 2\beta$$
$$+(a_2\cosh 2\alpha+a'_4\cosh 4\alpha)\cos 4\beta. \qquad\ldots\quad(19)$$

Then from (17) and (18) we have the following equations for the constants:

$$\left.\begin{aligned}a_0&=\frac{1}{32}\,\frac{2\lambda+3\mu}{\lambda+2\mu}\,\rho\omega^2c^4\cosh 2\alpha_0\\a_2&=\frac{1}{96}\,\frac{2\lambda+\mu}{\lambda+2\mu}\,\rho\omega^2c^4\,\frac{\cosh 2\alpha_0}{2+\cosh 4\alpha_0}\,.\\a'_2&=-\frac{1}{64}\rho\omega^2c^4\left[\frac{1}{3}\cdot\frac{2\lambda+\mu}{\lambda+2\mu}\cdot\frac{2+3\cosh 4\alpha_0}{2+\cosh 4\alpha_0}\right.\\&\qquad\left.+\frac{2\lambda+3\mu}{\lambda+2\mu}\right]\\a'_4&=-\frac{1}{384}\,\frac{2\lambda+\mu}{\lambda+2\mu}\,\rho\omega^2c^4\cdot\frac{1}{2+\cosh 4\alpha_0}\end{aligned}\right\}\quad\ldots\quad(20)$$

If the longitudinal tension be so adjusted that the length of the cylinder is maintained constant, we have $e=0$. If on the other hand the longitudinal extension be so adjusted that the tractions $\widehat{zz}$ at the ends have no statical resultant, we have

$$\int_0^{\alpha_0}\int_{\beta=0}^{\beta=\pi} \frac{\widehat{zz}}{h^2}\, d\alpha d\beta=0. \qquad \ldots \quad (21)$$

Substituting for $\widehat{zz}$ and simplifying we find

$$\frac{2\lambda}{(\lambda+\mu)c^2}\,\alpha_0+\frac{1}{2}\,\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,e-\frac{\lambda}{16(\lambda+2\mu)}\rho\omega^2c^2\cosh 2\alpha_0=0.$$

Therefore

$$e=-\frac{\lambda}{8\mu(3\lambda+2\mu)}\rho\omega^2c^2\cosh 2\alpha_0 \qquad \ldots \quad (22)$$

in agreement with the result obtained by Chree.

We have for $\frac{1}{h^2}\frac{\partial P}{\partial \alpha}$ and $\frac{1}{h^2}\frac{\partial P}{\partial \beta}$ the values

$$\left.\begin{aligned}
\frac{1}{h^2}\frac{\partial P}{\partial \alpha} &= \frac{\lambda\mu c^2}{2(\lambda+\mu)}e\sin 2\beta+\frac{2(\lambda+2\mu)}{\lambda+\mu}[(-\alpha_0\\
&\qquad +a_2\cosh 4\alpha)\sin 2\beta-a_2\cosh 2\alpha\sin 4\beta]\\
\frac{1}{h^2}\frac{\partial P}{\partial \beta} &= -\frac{\lambda\mu c^2}{2(\lambda+\mu)}e\sinh 2\alpha+\frac{2(\lambda+2\mu)}{\lambda+\mu}\\
&\qquad \times[\alpha_0\sinh 2\alpha+a_2\sinh 4\alpha\cos 2\beta\\
&\qquad\qquad -a_2\sinh 2\alpha\cos 4\beta]
\end{aligned}\right\} \ldots \quad (23)$$

II.

4. In the case of a rotating elliptic plate we assume a state of plane stress defined by

$$\left.\begin{aligned} X_z=Y_z=Z_z=0 \\ X_x+\tfrac{1}{2}\rho\omega^2r^2=\frac{\partial^2\chi}{\partial y^2}, \quad Y_y+\tfrac{1}{2}\rho\omega^2r^2=\frac{\partial^2\chi}{\partial x^2}, \\ X_y=-\frac{\partial^2\chi}{\partial x\partial y} \end{aligned}\right\} \quad \dots \quad (24)$$

Since there are body forces whose components are $-\rho\omega^2x$, $-\rho\omega^2y$, 0, we have as in Art. 92 (*iv*) Love's Theory of Elasticity (3rd ed.)

$$\left.\begin{aligned} \nabla^2X_x+\frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial x^2}&=-\frac{4(\lambda+\mu)}{\lambda+2\mu}\rho\omega^2 \\ \nabla^2Y_y+\frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial y^2}&=-\frac{4(\lambda+\mu)}{\lambda+2\mu}\rho\omega^2 \\ \nabla^2Z_z+\frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial z^2}&=-\frac{2\lambda}{\lambda+2\mu}\rho\omega^2 \\ \nabla^2Y_z+\frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial y\partial z}&=0 \end{aligned}\right\} \quad \dots \quad (25)$$

and two similar equations for Z_x, and X_y

where $\Theta=X_x+Y_y+Z_z$.

Proceeding with equations (25) as in Art. 145, Love's Theory of Elasticity (3rd ed.), we have

$$\chi=\chi_0+\chi_1z-\frac{\lambda}{2(3\lambda+2\mu)}\Theta_0z^2-\frac{\lambda}{2(\lambda+2\mu)}\rho\omega^2r^2z^2 \quad \dots \quad (26)$$

where

$$\left.\begin{aligned} \nabla_1^2\chi_1&=\text{a constant } k \text{ (say)}. \\ \nabla_1^4\chi_0&=\frac{\lambda+2\mu}{\lambda+\mu}\rho\omega^2 \\ \text{and} \quad \Theta_0&=\nabla_1^2\chi_0-\rho\omega^2r^2 \end{aligned}\right\} \quad \dots \quad (27)$$

If we put $\chi_0=\chi_0'+\frac{1}{512}\frac{\lambda+2\mu}{\lambda+\mu}\rho\omega^2c^4\,(\cosh 4\alpha+\cos 4\beta)$

then $$\nabla_1^4\chi_0'=0. \qquad \dots \quad (28)$$

The displacements are given by (14) where P satisfies (15) and

$$h^2\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{h^2}\frac{\partial P}{\partial\beta}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{h^2}\frac{\partial P}{\partial\alpha}\right)\right]$$

$$=\frac{4(\lambda+\mu)}{3\lambda+2\mu}\,(\nabla_1^2\chi_0'+kz)-\frac{2\lambda}{\lambda+2\mu}\rho\omega^2z^2. \qquad \dots \quad (29)$$

The terms in P dependent on z are given by

$$\left.\begin{aligned}\frac{1}{h^2}\frac{\partial P}{\partial\alpha}&=-\left[\frac{\lambda+\mu}{3\lambda+2\mu}kz-\frac{\lambda}{2(\lambda+2\mu)}\rho\omega^2z^2\right]c^2\sin 2\beta\\ \frac{1}{h^2}\frac{\partial P}{\partial\beta}&=\left[\frac{\lambda+\mu}{3\lambda+2\mu}kz-\frac{\lambda}{2(\lambda+2\mu)}\rho\omega^2z^2\right]c^2\sinh 2\alpha\end{aligned}\right\}\dots \quad (30)$$

Also since $\widehat{zz}=0$, $\widehat{\alpha z}=\widehat{\beta z}=0$, we can easily show that

$$w=\frac{\chi_1}{2\mu}-\frac{\lambda+\mu}{4\mu(3\lambda+2\mu)}kc^2\,(\cosh 2\alpha+\cos 2\beta)$$

$$-\frac{\lambda}{2\mu(3\lambda+2\mu)}\left[\Theta_0z+\tfrac{1}{2}kz^2-\frac{\lambda(3\lambda+2\mu)}{6(\lambda+\mu)(\lambda+2\mu)}\rho\omega^2z^3\right]. \qquad (31)$$

5. To apply the above solution to a plate whose boundary is given by $\alpha=\alpha_0$, we assume

$$\left.\begin{aligned}&\chi_1=0\\ &\chi_0'=a_0\cosh 2\alpha+(a_0+a_2'\cosh 2\alpha+a_2\cosh 4\alpha)\cos 2\beta\\ &\qquad+(a_4\cosh 2\alpha+a_4'\cosh 4\alpha)\cos 4\beta\end{aligned}\right\}\quad \dots \quad (32)$$

With the help of (26), (27) and (32) we can determine the stress system X_x, X_y and Y_y given by (24) or the equivalent stress system $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ and $\widehat{\beta\beta}$ given by the first three equations of (10).

Assuming that on the boundary $\alpha=\alpha_0$ we have

$$\int_{-l}^{l} \widehat{\alpha\alpha}\, ds=0 \qquad \text{and} \qquad \int_{-l}^{l} \widehat{\alpha\beta}\, ds=0$$

where $2l$ is the thickness of the plate, we have the following equations for the constants in (32).

$$\left.\begin{aligned}
a_0 &= \frac{1}{48}\,\frac{\lambda(3\lambda+2\mu)}{(\lambda+\mu)(\lambda+2\mu)}\,\rho\omega^2 c^2 l^2 \\
&\qquad + \frac{1}{128}\,\frac{7\lambda+6\mu}{\lambda+\mu}\,\rho\omega^2 c^4 \cosh 2\alpha_0 \\
a_2 &= \frac{1}{384}\,\frac{5\lambda+2\mu}{\lambda+\mu}\,\rho\omega^2 c^4\,\frac{\cosh 2\alpha_0}{2+\cosh 4\alpha_0} \\
a_2' &= \frac{1}{48}\,\frac{\lambda(5\lambda+2\mu)}{(\lambda+\mu)(3\lambda+2\mu)}\,\rho\omega^2 c^2 l^2\,\frac{\cosh 2\alpha_0}{2+\cosh 4\alpha_0} \\
&\qquad -\frac{1}{256}\,\rho\omega^2 c^4 \left[\frac{1}{3}\,\frac{5\lambda+2\mu}{\lambda+\mu}\,\frac{2+3\cosh 4\alpha_0}{2+\cosh 4\alpha_0} + \frac{7\lambda+6\mu}{\lambda+\mu}\right] \\
a_4' &= -\frac{1}{1536}\,\frac{5\lambda+2\mu}{\lambda+\mu}\,\rho\omega^2 c^4\,\frac{1}{2+\cosh 4\alpha_0}
\end{aligned}\right\} \quad \ldots \quad (34)$$

In conclusion I wish to express my thanks to Dr. N. R. Sen for guidance in course of the work.

Bull. Cal. Math. Soc., Vol. XIX, No. 3 (1928).

3

On the strong summability (C1) of the Fourier series of a function at a point where the function has an infinite discontinuity of the second kind

BY

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The object of the present paper is to formulate a definite answer to the following question :—Can the Fourier series of a function $f(x)$ be strongly summable (C1) at a point x_0, where $f(x)$ has an *infinite* discontinuity of the second kind such that

$$\phi(t) \equiv \frac{1}{2}\left\{ f(x_0+t)+f(x_0-t)-2s \right\} = \chi(t)\cos\psi(t),$$

$\chi(t)$ and $\psi(t)$ being both monotone functions which tend to infinity as t tends to o and s being the sum (C1) at x_0 ? The criterion, which Hardy and Littlewood* gave in 1927, is the most comprehensive of the various criteria published up to now, but it is *not satisfied* in the cases covered by the aforesaid question. Notwithstanding the failure of the criterion of Hardy and Littlewood to enable us to answer the question, I prove that, if

$$\psi \succ \log\frac{1}{t^2}, \quad \frac{\chi}{\psi'} \prec t, \text{ and, further, } \int_0^t \left\{ \chi(u) \right\}^2 du \prec t^{1-\lambda} \text{ where } 1>\lambda>0,$$

then there is strongly summability, and

$$\sum_{m=0}^{n} \left| s_m - s \right|^q = 0(n)$$

for every positive q.

* *Proc. L. M. S.*, Ser. 2, Vol. 26, pp. 278-287.

For the sake of simplicity, x_0 is taken to be 0, $f(t)$ is taken to be an even function and its mean value is taken to be zero.

Preliminary Results.

1. For the sake of convenience, the following results are quoted here from my previous papers, $\psi(t)$ being $\succ \log \frac{1}{t^2}$ and $\frac{\chi}{\psi'}$ being $\prec t$:—

(A)* The series is summable (C1) to 0.

(B)† $\int_0^\delta \phi(t)dt = O(\delta)$, but $\int_0^\delta | \phi(t) | \, dt$ has an infinite differential co-efficient at $\delta = 0$ and is therefore not even $O(\delta)$.

(C)‡ $\int_0^t \chi(t) \cos \psi(t) dt = \frac{\chi}{\psi'} \sin \psi + \frac{1}{\psi'} \frac{d}{dt}\left(\frac{\chi}{\psi'}\right) \cos \psi + \ldots\ldots,$

so that for small values of t the integral behaves as $\frac{\chi}{\psi'} \sin \psi$.

Question answered.

2. I proceed to prove the truth of the answer, *viz.*, that, if

$$\psi \succ \log \frac{1}{t^2}, \quad \frac{\chi}{\psi'} \prec t, \int_0^t \left\{ \chi(u) \right\}^2 du \prec t^{1-\lambda},$$

where λ is a positive proper fraction, then

$$\sum_{m=0}^{n} \left| s_m - s \right|^q = O(n)$$

for every positive q.

* *Bulletin of the Calcutta Mathematical Society*, Vol. XIX, pp. 51-58.
† *Ibid*, pp. 1-12.
‡ Ibid, foot note of p. 53.

Proof: *

Following the previous writers, let

$$\pi s_n = \int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t} f(t)$$

$$= \int_0^{\frac{k}{n}} \sin nt. \cot \tfrac{1}{2}t.f(t)dt + \int_{\frac{k}{n}}^\pi \sin nt. \cot \tfrac{1}{2}t.f(t)dt$$

$$+ \int_0^\pi \cos nt. f(t)dt,$$

where k is fixed beforehand, and let the above three integrals be respectively denoted by α_n, β_n, γ_n.

By Minkowski's inequality

$$\pi \left(\sum_1^n | s_m |^q \right)^{\frac{1}{q}} \leq \left(\sum_1^n | \alpha_m |^q \right)^{\frac{1}{q}} + \left(\sum_1^n | \beta_m |^q \right)^{\frac{1}{q}}$$

$$+ \left(\sum_1^n | \gamma_m |^q \right)^{\frac{1}{q}}, \qquad \text{(I)}$$

and I proceed to consider the three sums separately.

(*a*) Let τ stand for $\frac{k}{n}$ and $F(t)$ for $\int_0^t f(u)du$. Then, integrating by parts,

* This proof is based on the proofs of Carleman (*Proc. L. M. S.*, Ser. 2, Vol. 21), Sutton (*Proc. L. M. S.*, Ser. 2, Vol. 23), and Hardy and Littlewood (*l.c.*).

$$a_m = \sin m\tau \cot \tfrac{1}{2}\tau \; F(\tau)$$

$$-\int_0^\tau \left(m \cos mt \cot \frac{t}{2} - \frac{1}{2} \sin mt \operatorname{cosec}^2 \tfrac{1}{2}t\right) F(t)dt.$$

Now, according to (C) of Art. 1, $F(\tau)$ behaves as $\frac{\chi(\tau)}{\psi'(\tau)}$ which, again, by hypothesis $\prec \tau$. Therefore, for small values of τ,

$$\sin m\tau \cot \tfrac{1}{2}\tau \; F(\tau) = O(1).$$

Also $\left| m \int_0^\tau \cos mt \cot \tfrac{t}{2} F(t)\, dt \right| < \pi m \int_0^\tau \frac{1}{t} \frac{\chi(t)}{\psi'(t)} dt = O(1),$

and $\left| \tfrac{1}{2} \int_0^\tau \sin mt \operatorname{cosec}^2 \tfrac{1}{2}t . F(t)\, dt \right| < \pi^2 \int_0^\tau \left| \frac{\sin mt}{t} \right| . \frac{1}{t} \frac{\chi(t)}{\psi'(t)} dt$

$$< \pi^2 m \int_0^\tau \frac{1}{t} \frac{\chi}{\psi'} dt = O(1).$$

Thus we have

$$a_m = O(1),$$

and, consequently, for $n >$ a number $n_1(\epsilon, k)$ dependent on ϵ and k,

$$\left(\sum_0^n \left| a_m \right|^q \right)^{\frac{1}{q}} < \epsilon . n^{\frac{1}{q}}. \qquad (1)$$

(b) $\beta_m = \int_\tau^\pi \sin mt \cot \tfrac{1}{2} t . f(t)\, dt$

$$= \int_\tau^\pi \cot \tfrac{1}{2} t \left(\frac{d}{dt} \int_0^t \sin mu \; f(u)\, du \right) dt$$

$$=-\tfrac{1}{2}\,\pi\cot\tfrac{1}{2}\,\tau.\;c_m(\tau)+\tfrac{1}{4}\,\pi\int_\tau^\pi \operatorname{cosec}^2\tfrac{1}{2}\,t.\;c_m(t)\,dt \tag{2}$$

by integrating by parts, $c_m(s)$ standing for the mth Fourier sine coefficient of the odd function $f_1(t)$ which is equal to $f(t)$ in (o, s) and to zero in (s, π).

Applying Minkowski's inequality to (2) we have

$$\Big(\sum_1^n|\beta_m|^q\Big)^{\frac{1}{q}}\le\frac{1}{2}\,\pi\cot\frac{1}{2}\,\tau\,\Big(\sum_1^n|c_m(\tau)|^q\Big)^{\frac{1}{q}}$$

$$+\frac{1}{4}\,\pi\int_\tau^\pi\operatorname{cosec}^2\frac{1}{2}\,t\Big(\sum_1^n|c_m(t)|^q\Big)^{\frac{1}{q}}dt \qquad \dots\ (3)$$

But, by Hausdorff's inequality,*

$$\Big(\sum_1^n|c_m(t)|^q\Big)^{\frac{1}{q}}\le\Big(\frac{1}{\pi}\int_{-\pi}^\pi|f_1(u)|^{q'}\,du\Big)^{\frac{1}{q'}}$$

$$=\Big(\frac{1}{\pi}\int_{-t}^t|f(u)|^{q'}\,du\Big)^{\frac{1}{q'}},$$

q' standing for $\frac{q}{q-1}$ and q being assumed to be greater than 2.

* If a_k be the typical Fourier constant of a function $f(x)$ whose rth power summable and if $\frac{1}{r}+\frac{1}{s}=1$, $s>r$, then

$$\Big[\sum_{k=0}^\infty|a_k|^s\Big]^{\frac{1}{s}}\le\Big[\frac{1}{\pi}\int_{-\pi}^\pi|f(t)|^r\Big]^{\frac{1}{r}}.$$

(*Math. Zeit.*, Bd. 16, pp. 163-169).

But $|f(u)| \leq \chi(u)$ for every small value of u. Therefore

$$\int_{-t}^{t} |f(u)|^{q'}\,du = 2\int_{0}^{t} |f(u)|^{q'}\,du < 2\int_{0}^{t} \{\chi(u)\}^{q'}\,du.$$

Now χ is unbounded and monotone ; consequently

$$\{\chi(u)\}^{q'} < \{\chi(u)\}^{2}$$

as $q' < 2$. Therefore

$$\int_{0}^{t} \{\chi(u)\}^{q'}\,du < \int_{0}^{t} \{\chi(u)\}^{2}\,du < t^{1-\lambda} \text{ by hypothesis.}$$

Therefore

$$\left(\sum_{1}^{n} |c_n(t)|^{q}\right)^{\frac{1}{q}} < A\, t^{\frac{1}{q'}-\frac{\lambda}{q'}}$$

where A is a constant.

Using this inequality in (3), we have, for small values of τ,

$$\left(\sum_{1}^{n} |\beta_n|^{q}\right)^{\frac{1}{q}} < \left(\frac{\pi}{2}\right)^{2} \cdot \frac{A\,\tau^{\frac{1}{q'}-\frac{\lambda}{q'}}}{\frac{\tau}{2}} + \frac{A}{4}\,\pi\left(\frac{\pi}{2}\right)^{2}\int_{\tau}^{\pi} \frac{t^{\frac{1}{q'}-\frac{\lambda}{q'}}}{\frac{t^{2}}{4}}\,dt$$

$$< B.\,\tau^{\frac{1}{q'}-1-\frac{\lambda}{q'}}, \text{ i.e., } B\,\tau^{-\frac{1}{q}-\frac{\lambda}{q'}},$$

where B is a constant.

Thus it is proved that

$$\left(\sum_{1}^{n} |\beta_n|^{q}\right)^{\frac{1}{q}} < \epsilon_k\, n^{\frac{1}{q}+\frac{\lambda}{q'}} \qquad \ldots \quad (4)$$

where ϵ_k is a function of k only which tends to 0 when $k \longrightarrow \infty$.

(c) It is obvious that $\gamma_n \longrightarrow 0$, so that

$$\left(\sum_1^n |\gamma_n|^q\right)^{\frac{1}{q}} < \epsilon n^{\frac{1}{q}} \qquad \cdots \quad (5)$$

for $n>$ a number $n_2(\epsilon)$ dependent on ϵ.

(d) Using (1), (4) and (5) in (I), we have

$$\pi \left(\sum_1^n |s_m|^q\right)^{\frac{1}{q}} < (\epsilon_k + 2\epsilon)\, n^{\frac{1}{q}+\frac{\lambda}{q'}} \text{ if } n \text{ is greater than } n_1 \text{ and } n_2.$$

Since $\frac{1}{q}+\frac{\lambda}{q'} < 1$, by choice first of k and then of n, it is proved that

$$\left(\sum_1^n |s_m|^q\right)^{\frac{1}{q}} = o(n).$$

Failure of the criterion of Hardy and Littlewood.

3. The criterion of Hardy and Littlewood has been stated by them as follows:

"If $p>1$ and $\int_0^t |\phi(u)|^p\, du = O(t)$, $\int_0^t \phi(u)du = o(t)$,

then $\sum_0^n |s_n - s|^q = o(n)$ (for every positive q."

Now it follows from (B) of Art. 1, that, whilst in the cases under consideration

$$\int_0^t \phi(u)du = o(t),$$

$$\int_0^t |\phi(u)|\, du \succ t.$$

Thus it is obvious that the first condition in the criterion can never be satisfied. Hardy and Littlewood assume (*l.c.* p. 277) that if this condition is satisfied for any value of p it will be satisfied by all higher values. But as p may be as near as we please to **1**, it is obvious that the condition can never be satisfied for any $p > 1$.

Notwithstanding the fact that the first condition of the criterion is not satisfied, there is strong summability as proved in the preceding article. Therefore the failure of the criterion of Hardy and Littlewood has been established.

Conclusion.

4. In their paper, Hardy and Littlewood say of their criterion that it "has more the air of finality" than the criteria of Carleman and Sutton. But, in reality, the problem of the strong summability (C1) at a given point is of the same class as the problem of ordinary convergence or ordinary summability, and there is as little chance of a stage of finality having been reached in reference to any one of these problems as in reference to the other two

I refrain from giving certain obvious generalizations of the result proved in Art. 2.

I have proved * that a function can be found the Fourier series of which may be ordinarily summable (C1 at a given point even if

$$\int_0^t \phi(u)du \neq O(t).$$

It is likely that for some of such functions there may be even strong summability.

* *Bulletin of the Calcutta Mathematical Society*, Vol. XIX, pp. 51-58.

4

NOTE ON THE DIFFERENCE EQUATIONS DEFINING ENUMERATIVE ARITHMETICAL FUNCTIONS

BY

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1. If $f(x)$ takes a single finite value for each integral value $\gtreqless 0$ of x, we call $f(x)$ an arithmetical function of x. For example, $N_p(x)$, the number of representations of n as the sum of p integral squares, with roots $\gtreqless 0$, the arrangements of the squares in a given representation being taken into account, is an arithmetical function of x.

Many of the most interesting arithmetical functions in the literature are definable by linear difference equations, and conversely, the occurrence of a function in one of these linear difference equations serves to define it. The simplest identities between elliptic theta constants or functions provide us with an inexhaustible store of such difference equations and their solutions. We shall illustrate these by two instances, the first of which gives a fundamental formula relating to the number of representations of an integer as the sum of an even number of squares.

2. In his note 'sur le nombre des representations des nombres par une somme d'un nombre pair de carrés,'* J. V. Ouspensky gives the following theorem. If $\Phi(n)$ is an arithmetical function defined for $n=0, 1, 2, \ldots$, and if $\Phi(0)=1$,

$$(1)\quad \Sigma[n-(p+1)j^2]\Phi(n-j^2)=0, \qquad (j=0, \pm 1, \pm 2, \ldots),$$

* *Bulletin de l'Académie des Sciences de l'URSS*, 1925, p. 647.

the sum referring to j and continuing so long as $n-j^2 \geqq 0$, then $\Phi(n)=N_p(n)$. Since this difference equation determines $\Phi(k)$ $(k=1, 2,\ldots)$ successively and uniquely from $\Phi(0)$, it is sufficient to prove that $N_p(n)$ satisfies the same equation, namely

$$(2)\qquad \Sigma[n-(p+1)j^2]N_p(n-j^2)=0.$$

Professor Ouspensky gives a purely arithmetical proof of the last. An algebraic proof, however, more readily suggests the means by which such theorems may be written down at will. We give such a proof next.

3. If, in the usual notation,

$$\theta_3(q)\equiv\theta_3=\Sigma q^{j^2}=1+2q+2q^4+2q^9+2q^{16}+\ldots,$$

we have

$$\theta_3^p=\Sigma q^n N_p(n),$$

the Σ referring to $n=0, 1, 2,\ldots$, and hence

$$\theta_3^{p+1}=\Sigma q^n N_p(n)\times\Sigma q^{j^2},$$

$$=\Sigma q^n[\Sigma N_p(n-j^2)],$$

the second Σ referring to $j=0, \pm1, \pm2,\ldots$, and continuing so long as $j^2\leqq n$, as before. To both sides of the last identity apply the operation $q\dfrac{d}{dq}$. Then

$$(p+1)\theta_3^p q\frac{d}{dq}\theta_3=\Sigma q^n n[\Sigma N_p(n-j^2)],$$

the left of which is

$$(p+1)\Sigma q^n N_p(n)\times\Sigma j^2 q^{j^2}$$

$$=(p+1)\Sigma q^n[\Sigma j^2 N_p(n-j^2)].$$

Hence, by equating coefficients of q^n we get (2). The origin of (2) therefore appears in the obvious remark that an identity remains an identity after differentiation. Clearly the like can be applied indefinitely.

4. It will be sufficient to state the next, as its proof is by the same device of differentiating two trivial identities, namely

$$\theta_2^{r+1}(q^4)\theta_3^s(q^4)=\theta_2(q^4)\times\theta_2^r(q^4)\theta_3^s(q^4),$$

$$\theta_2^r(q^4)\theta_3^{s+1}(q^4)=\theta_3(q^4)\times\theta_2^r(q^4)\theta_3^s(q^4),$$

where

$$\theta_2(q^4)=\Sigma q^{h^2}\ (h=\pm1,\pm3,\ \pm5,\ldots).$$

Let $N(n;\ r,\ s)$ denote the number of representations of n as the sum of $r+s$ integral squares, with roots $\gtreqless 0$, precisely r of which are odd and occupy the first r places in each of the representations. Then

$$\theta_2^r(q^4)\theta_3^s(q^4)=\Sigma q^{4n+r}N(4n+r;\ r,\ s),$$

the Σ referring to $n=0,1,2,\ldots$. Proceeding as indicated, we get the pair of linear difference equations, which, with the obvious initial conditions, uniquely define $N(n;\ r,\ s)$,

$$(3)\quad \Sigma[4n+r+1-(r+1)h^2]N(4n+r+1-h^2\ ;\ r,s)$$
$$=s\Sigma k^2N(4n+r+1-k^2;\ r+1,\ s-1),$$

$$(4)\quad \Sigma[4n+r-(s+1)h^2]\ N(4n+r-k^2;\ r,\ s)$$
$$=r\Sigma h^2\ N(4n+r-h^2;\ r-1,\ s+1),$$

the Σ's referring to $h=\pm1,\ \pm3,\ \pm5,\ldots,\quad k=0,\ \pm2,\ \pm4,\ \pm6,\ldots$, and continuing so long as all arguments of N are ≥ 0.

In (4) replace r by $r+1$, s by $s-1$. Then, by elimination between the result and (3), we find

$$\Sigma N(4n+r+1-k^2;\ r+1,\ s-1)=\Sigma N(4n+r+1-h^2;\ r,\ s),$$

which is equivalent to the identity

$$\theta_2^{r+1}\ (q^4)\theta_3^{s-1}\ (q^4)\times\theta_3(q^4)=\theta_2^r\ (q^4)\theta_3^s\ (q^4)\times\theta_2(q^4),$$

thus providing a check.

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ON THE SOLUTION OF THE GENERAL ALGEBRAIC EQUATION IN A SYMMETRICAL FORM

BY

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1. In a recent paper * I have given a generalization of the well-known inversion formula of Lagrange. The new formula, obtained by me, contains within itself the solution in series of any equation, algebraic or transcendental. When an equation admits of being transformed into the form (1) considered below, the solution in series, as directly obtained from the formula, has got an elegant and compact presentation, which it is the object of this paper to elucidate.

2. Starting then with an equation expressed in the form

$$z^m = \omega^m + z^p\,(a_0 + a_1 z + a_2 z^2 + \ldots + a_q z^q) \qquad \ldots \quad (1)$$

where m and q are positive integers and p is in general a fraction, let us consider a region within the circle C enclosing a root ω of the equation $z^m - \omega^m = 0$. Suppose that the co-efficients $\{ a_\lambda \}$ of the equation (1) are such that along the entire circumference of the circle C the condition

$$| z^m - \omega^m | > | z^p (a_0 + a_1 z + a_2 z^2 + \ldots + a_q z^q) |$$

is satisfied; then the equation (1) has a single root ξ within the circle.

Now writing $(a_0 + a_1 z + a_2 z^2 + \ldots + a_q z^q)^r$ in the expanded form

$${}_rA_0 + {}_rA_1 z + {}_rA_2 z^2 + \ldots\ldots + {}_rA_s z^s + \ldots\ldots + {}_rA_{qr} z^{qr} \dagger$$

and by applying the formula (1) of my previous paper

* *Bull. Cal. Math. Society*, Vol. XIX, No. 1, pp. 21-24.

† For the calculation of the co-efficients ${}_rA_s$ see my paper in Vol. XIV, No. 3, of this *Bulletin*.

$\psi(\xi)$ is given by the convergent series

$$\psi(\omega)+\sum_{r=1}^{\infty}\left[\frac{1}{r!}\left(\frac{1}{m\omega^{m-1}}\frac{d}{d\omega}\right)^{r-1}\frac{\omega^{-m+1}}{m}\psi'(\omega)\sum_{s=0}^{qr}{}_rA_s\omega^{pr+s}\right].$$

Choosing $\psi(\xi)$ such that $\psi'(\xi)=\xi^{\nu-1}$, where ν is an arbitrary constant we can express the above as

$$\psi(\xi)=\psi(\omega)+\sum_{r=1}^{\infty}\left[\frac{1}{r!}\left(\frac{1}{m\omega^{m-1}}\frac{d}{d\omega}\right)^{r-1}\frac{1}{m}\sum_{s=0}^{qr}{}_rA_s\omega^{pr+s-m+\nu}\right]$$

or, $$\psi(\omega)+\sum_{r=1}^{\infty}\left[\frac{1}{r!m}\left(\frac{1}{m\omega^{m-1}}\frac{d}{d\omega}\right)^{r-2}\sum_{s=0}^{qr}\left(\frac{pr+s+\nu}{m}-1\right)\times{}_rA_s\omega^{pr+s-2m+\nu}\right],$$

the ultimate form being

$$\psi(\xi)=\psi(\omega)+\sum_{r=1}^{\infty}\frac{1}{r!m}\sum_{s=0}^{qr}{}_r\rho_s\;{}_rA_s\omega^{pr+s-rm+\nu} \quad \ldots \quad (2)$$

where

${}_r\rho_s$ for $r>1$ denotes the number

$$\left(\frac{pr+s+\nu}{m}-1\right)\left(\frac{pr+s+\nu}{m}-2\right)\ \ldots\ \left\{\frac{pr+s+\nu}{m}-(r-1)\right\}$$

and for $r=1$ is equivalent to 1.

3. Consider now a term

$$\frac{1}{r!m}\sum_{s=0}^{qr}{}_r\rho_s\cdot{}_rA_s\omega^{pr+s-rm+\nu} \quad \ldots \quad (3)$$

from (2) above. Let us choose a quantity r' such that

$$(r'+1)\,m-1 \nless qr.$$

If now the upper limit of summation in (3) be extended to $(r'+1)\,m-1$, the value of the term will remain the same, because

${}_rA_s=0$ when $s>qr$. The expression (3) can then be written in a compact form

$$\frac{1}{r!m}\sum_{s=0}^{m-1}{}_r\mu_s\ \omega^{pr+s+\nu} \qquad \dots \quad (4)$$

where ${}_r\mu_s$ stands for

$$\omega^{-rm}\left({}_r\rho_s\ {}_rA_s+\omega^{m}\ {}_r\rho_{s+m}\ {}_rA_{s+m}+\omega^{2m}\ {}_r\rho_{s+2m}\ {}_rA_{s+2m}+\dots\right.$$
$$\left.+\omega^{rm'}{}_r\rho_{s+rm'}\ {}_rA_{s+rm'}\right).$$

Treating each of the other terms in the same manner, we can express (2) in the form

$$\psi(\xi)=\psi(\omega)+\sum_{r=1}^{\infty}\frac{1}{r!m}\sum_{s=0}^{m-1}{}_r\mu_s\omega^{pr+s+\nu} \qquad \dots \quad (5)$$

4. Let t be the least positive integral number so chosen that the product $t\times p$ is an integer. Form now the function

$$\frac{{}_r\mu_s}{r!}+\frac{{}_{r+t}\mu_{s-pt}}{(r+t)!}+\frac{{}_{r+2t}\mu_{s-2pt}}{(r+2t)!}+\dots\dots+\frac{{}_{r+kt}\mu_{s-kpt}}{(r+kt)!}+\dots$$

denoted by ${}_r\gamma_s$, where the suffixes $(s-kpt)$ are all to lie between 0 and $m-1$. If for any integer k, $s-kpt<0$ or $>m-1$, then we replace in the above the $(k+1)^{th}$ term by

$$\omega^{\pm lm}\frac{{}_{r+kt}\mu_{s-kpt\pm lm}}{(r+kt)!},$$

where l is to be such an integer that $s-kpt\pm lm$ lies within 0 and $m-1$.

${}_r\gamma_s$ being defined in this manner, the series (5) after further grouping of terms may be expressed in the form

$$\psi(\xi)=\psi(\omega)+\frac{1}{m}\sum_{r=1}^{t}\sum_{s=0}^{m-1}{}_r\gamma_s\omega^{pr+s+\nu} \qquad \dots \quad (6)$$

There is no loss of generality if we assume $\omega^m=1$. The functions ${}_r\gamma_s$ are then also involved in similar expressions for the remaining roots of the equation (1). The importance of these functions

in the complete solution of the equation is therefore evident. In general, these functions will be convergent for sufficiently small values of $|a_\lambda|$.

If p is an integer t is equal to 1 and the formula (6) reduces to

$$\psi(\xi)=\psi(\omega)+\frac{1}{m}\sum_{s=0}^{m-1}{}_1\gamma_s\,\omega^{p+s+\nu} \qquad \ldots \quad (7)$$

5. The equation (1) when p is an integer has been used as a basis for the solution of the general algebraic equation by means of series. In the year 1895, McClintock * gave a method of calculating all the roots of an algebraic equation by transforming it in different ways to this standard form, securing by each such transformation a certain number of unequal roots of the original equation. The formula (7) of this paper serves to express these roots in a symmetrical and compact form.

In a number of papers Prof. Birkeland † has considered the problem of presenting the roots of the general algebraic equation by means of known functions. Starting from a standard form akin to (1) his analysis has enabled him to express its roots as a linear sum of a certain number of hypergeometric series of many variables.

It may be noticed that when an equation is particularly rich in nearly equal roots the standard form (1) when p is an integer becomes unsuitable.

My best thanks are due to Prof. Ganesh Prasad for his kind interest and encouragement.

* *American Journal of Mathematics*, Vol. XVII, pp. 89-110.

† 'Resolution de l'equation algebrique generale par des fonctions hypergeometriques de plusieurs variables,' *C.R.* Vol. 171. 'Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen,' *Math. Zeit.*, Band 26, Heft 4

6

ON THE EXPANSION OF θ IN THE MEAN VALUE THEOREM OF THE DIFFERENTIAL CALCULUS

BY

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1. The object of the present paper is to show how the general term in the expansion,

$$\theta = \sum_{0}^{\infty} A_n h^n \tag{1}$$

can be obtained, θ being the function which occurs in the *mean value theorem* of the Differential Calculus, *i.e.*, the theorem

$$f(x+h)=f(x)+hf_1(x+\theta h). \tag{2}$$

In Edwards's* "Treatise on the Differential Calculus," A_0, A_1, A_2, A_3 have been given, and Whitcom† gave also the values of A_4 and A_5. I give the values of two more co-efficients A_6 and A_7, and, further, I give the general linear equation connecting $A_0, A_1, A_2, \ldots A_n$, which, when solved simultaneously with $(n-1)$ similar equations, gives A_n and the preceding co-efficients.

It is believed that my results are new.

* Joseph Edwards : "An Elementary Treatise on the Differential Calculus", pp. 103, 110.

† A. W. Whitcom : "On the expansion of $\phi(x+h)$," *American Journal of Mathematics*, Vol. III.

2. The known co-efficients are

$$A_0=\frac{1}{2},$$

$$A_1=\frac{1}{24}\frac{f_3}{f_2},$$

$$A_2=\frac{1}{\lfloor 2}\left[\frac{1}{24}\frac{f_4}{f_2}-\frac{1}{24}\frac{f_3^2}{f_2^2}\right],$$

$$A_3=\frac{1}{\lfloor 3}\left[\frac{11}{320}\frac{f_5}{f_2}-\frac{3}{32}\frac{f_3f_4}{f_2^2}+\frac{11}{192}\frac{f_3^3}{f_2^3}\right],$$

$$A_4=\frac{1}{\lfloor 4}\left[\frac{13}{480}\frac{f_6}{f_2}-\frac{43}{480}\frac{f_3f_5}{f_2^2}+\frac{7}{32}\frac{f_3^2f_4}{f_2^3}\right.$$

$$\left.-\frac{1}{16}\frac{f_4^2}{f_2^2}-\frac{3}{32}\frac{f_3^4}{f_2^4}\right],$$

$$A_5=\frac{1}{\lfloor 5}\left[\frac{19}{896}\frac{f_7}{f_2}-\frac{31}{384}\frac{f_3f_6}{f_2^2}+\frac{15}{64}\frac{f_3^2f_5}{f_2^3}-\frac{55}{108}\frac{f_3^3f_4}{f_2^4}\right.$$

$$\left.+\frac{5}{16}\frac{f_3f_4^2}{f_2^3}-\frac{53}{384}\frac{f_4f_5}{f_2^2}+\frac{185}{1152}\frac{f_3^5}{f_2^5}\right],$$

where $f_2, f_3, f_4,$...stand for $f_2(x), f_3(x), f_4(x)$..., the second, third, fourth,...derivatives of $f(x)$.

3. Expanding (2) by Taylor's theorem and putting the value of θ as in (1) and then comparing the co-efficients of different powers of h, we get a series of simultaneous equations. Comparing the co-efficients of h^{n+2}, we have

$$A_nf_2+A_0A_{n-1}f_3+A_1A_{n-2}f_3+A_2A_{n-3}f_3+\ldots\ldots$$

$$+\frac{1}{\lfloor 2}A_0^2\ A_{n-2}f_4+A_0A_1A_{n-3}f_4+A_0A_2A_{n-4}f_4+\ldots\ldots$$

$$+\frac{1}{\lfloor 3}A_0^3\ A_{n-3}f_5+\frac{1}{\lfloor 2}A_0^2\ A_1A_{n-4}f_5$$

$$+\frac{1}{\lfloor 2}A_0^2\ A_2A_{n-5}f_5+\ldots\ldots$$

$$+\frac{1}{\underline{|4}}A_0^4A_{n-4}f_6+\frac{1}{\underline{|3}}A_0^3A_1A_{n-5}f_6$$

$$+\frac{1}{\underline{|3}}A_0^3\ A_2A_{n-6}f_6+\dots$$

$$+\frac{1}{\underline{|r}}A_0^r\ A_{n-r}\,f_{r+2}+\frac{1}{\underline{|r-1}}A_0^{r-1}\ A_1A_{n-r-1}\,f_{r+2}$$

$$+\frac{1}{\underline{|r-1}}A_0^{r-1}\ A_2A_{n-r-2}\,f_{r+2}\ +\dots$$

$$+\frac{1}{\underline{|n-4}}\ A_0^{n-4}\ \ A_4f_{n-2}+\frac{1}{\underline{|n-5}}\ A_0^{n-5}A_1A_3f_{n-2}$$

$$+\frac{1}{\underline{|2}}\ \frac{1}{\underline{|n-5}}\ A_0^{n-5}\ \ A_2^2\ f_{n-2}$$

$$+\frac{1}{\underline{|2}}\frac{1}{\underline{|n-6}}\ A_0^{n-6}\ \ A_1^2A_2\ f_{n-2}+\frac{1}{\underline{|4}}\ \frac{1}{\underline{|n-7}}\ A_0^{n-7}\ \ A_1^4f_{n-2}$$

$$+\frac{1}{\underline{|n-3}}\ A_0^{n-3}\ \ A_3\,f_{n-1}+\frac{1}{\underline{|n-4}}\ A_0^{n-4}\ \ A_1A_2f_{n-1}$$

$$+\frac{1}{\underline{|3}}\ \frac{1}{\underline{|n-5}}\ A_0^{n-5}A_1^3\ f_{n-1}$$

$$+\frac{1}{\underline{|n-2}}\ A_0^{n-2}\ A_2f_n+\frac{1}{\underline{|2}}\ \frac{1}{\underline{|n-3}}\ A_0^{n-3}\ \ A_1^2f_n$$

$$+\frac{1}{\underline{|n-1}}\ A_0^{n-1}\ \ A_1\,f_{n+1}+\frac{1}{\underline{|n+1}}\ A_0^{n+1}\ \ f_{n+2}=\frac{1}{\underline{|n+2}}\ f_{n+2}$$

This equation, solved simultaneously with $(n-1)$ similar equations, that are obtained by comparing the co-efficients of h^2, h^3, h^4,......h^{n+1}, will give the value of A_n in terms of the derivatives $f_2(x)$, $f_3(x)$, $f_4(x)$,......f_{n+2} (x). It is easy to see that these $(n-1)$ equations can be obtained from the given equation by giving different values to n.

4. Thus we have

$$A_6=\frac{1}{\underline{|6}}\left[\frac{15}{896}\frac{f_8}{f_2}-\frac{1}{14}\frac{f_3f_7}{f_2^2}+\frac{15}{64}\frac{f_3^2f_6}{f_2^3}-\frac{331}{576}\frac{f_3^3f_5}{f_2^4}\right.$$

$$+\frac{1175}{1152}\frac{f_3^4f_4}{f_2^5}-\frac{205}{192}\frac{f_3^2f_4^2}{f_2^4}+\frac{5}{32}\frac{f_4^3}{f_2^3}+\frac{47}{64}\frac{f_3f_4f_5}{f_2^3}-\frac{11}{128}\frac{f_5^2}{f_2^2}$$

$$\left.-\frac{9}{64}\frac{f_4f_6}{f_2^2}-\frac{85}{384}\frac{f_3^6}{f_2^6}\right],$$

$$A_7=\frac{1}{\underline{|7}}\left[\frac{247}{18432}\frac{f_9}{f_2}-\frac{97}{1536}\frac{f_3f_8}{f_2^2}+\frac{703}{3072}\frac{f_3^2f_7}{f_2^3}-\frac{5551}{9216}\frac{f_3^3f_6}{f_2^4}\right.$$

$$+\frac{55363}{55296}\frac{f_3^4f_5}{f_2^5}-\frac{15995}{27648}\frac{f_3^5f_4}{f_2^6}+\frac{4375}{2304}\frac{f_3^3f_4^2}{f_2^5}$$

$$-\frac{245}{768}\frac{f_3f_4^3}{f_2^4}+\frac{287}{512}\frac{f_4^2f_5}{f_2^3}-\frac{8253}{3072}\frac{f_3^2f_4f_5}{f_2^4}$$

$$+\frac{14119}{30720}\frac{f_3f_5^2}{f_2^3}+\frac{35}{64}\frac{f_3f_4f_6}{f_2^3}-\frac{595}{3072}\frac{f_5f_6}{f_2^2}$$

$$\left.-\frac{71}{512}\frac{f_4f_7}{f_2^2}-\frac{2695}{18432}\frac{f_3^7}{f_2^7}\right].$$

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7

On certain modular equations and complex multiplication moduli

BY

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1. The object of the present note is (*i*) to determine the modular equations of the 131st and 151st orders and (*ii*) to deduce the complex multiplication moduli for the determinants $\Delta=135$, 147 and 363.* The procedure adopted by me is the same as that of Russell.†

The notation used is the following:—

k=modulus of the elliptic function,

$k'=\sqrt{1-k^2}$,

λ=modulus of the transformed elliptic function,

$\lambda'=\sqrt{1-\lambda^2}$,

$$P=\sqrt[4]{k\lambda}+\sqrt[4]{k'\lambda'}-1,$$

$$Q=\sqrt[4]{k\lambda k'\lambda'}-\sqrt[4]{k\lambda}-\sqrt[4]{k'\lambda'},$$

$$R=-\sqrt[4]{k\lambda k'\lambda'},$$

* According to Pascal's *Repertorium der Math.* (latest edition) and the list given by M. Hanna (*Proc. L. M. S.*, Series 2, Vol. 28, April, 1928) the modular equations of the 131st and 151st orders have not been given by any previous writer. Similarly according to the late Sir George Greenhill (*Proc. L. M. S.*, Series 1, Vols. 19 and 21) the complex multiplication moduli for the determinants $\Delta=135$ and 363 have not been given by any previous writers, while the class-invariant under $\Delta=147$ has been given by Berwick (*Proc. L. M. S.*, Series 2, Vol. 28, April, 1928) but his method is entirely different from that of mine.

† *Proc. L. M. S.*, Series 1, Vol. 21.

$$P_1 = \sqrt{k\lambda} + \sqrt{k'\lambda'} - 1,$$

$$Q_1 = \sqrt{k\lambda k'\lambda'} - \sqrt{k\lambda} - \sqrt{k'\lambda'},$$

$$R_1 = -\sqrt{k\lambda k'\lambda'}.$$

2. *Modular Equation of the 131st order.*

The modular equation can be written in the form

$$\begin{aligned}
&P_1^{11} + (16R_1)^{\frac{1}{3}}(a_1P_1^{10} + a_2P_1^8Q_1 + a_3P_1^6Q_1^2 + a_4P_1^4Q_1^3 + a_5P_1^2Q_1^4 \\
&+ a_6Q_1^5) + (16R_1)^{\frac{2}{3}}(b_1P_1^9 + b_2P_1^7Q_1 + b_3P_1^5Q_1^2 + b_4P_1^3Q_1^3 \\
&+ b_5P_1Q_1^4) + (16R_1)(c_1P_1^8 + c_2P_1^6Q_1 + c_3P_1^4Q_1^2 + c_4P_1^2Q_1^3 + c_5Q_1^4) \\
&+ (16R_1)^{\frac{4}{3}}(d_1P_1^7 + d_2P_1^5Q_1 + d_3P_1^3Q_1^2 + d_4P_1Q_1^3) + (16R_1)^{\frac{5}{3}}(e_1P_1^6 \\
&+ e_2P_1^4Q_1 + e_3P_1^2Q_1^2 + e_4Q_1^3) + (16R_1)^2(f_1P_1^5 + f_2P_1^3Q_1 + f_3P_1Q_1^2) \\
&+ (16R_1)^{\frac{7}{3}}(g_1P_1^4 + g_2P_1^2Q_1 + g_3Q_1^2) + (16R_1)^{\frac{8}{3}}(e'P_1^3 + e'_2P_1Q_1) \\
&+ (16R_1)^3(f_1'P_1^2 + f_2'Q_1) + (16R_1)^{\frac{10}{3}}(g_1'P_1) + (16R_1)^{\frac{11}{3}}h_1 = 0.
\end{aligned}$$

(*a*) If we put $\sqrt{\lambda} = \sqrt{k'}$, $\sqrt{\lambda'} = \sqrt{k}$ we obtain an equation in $z = (4kk')^{\frac{1}{6}}$ corresponding to $\omega = \sqrt{-131}$, $\sqrt{-115}$ and $\sqrt{-67}$.

(*b*) If we put $\sqrt{\lambda} = -\sqrt{k'}$, $\sqrt{\lambda'} = \sqrt{k}$, we obtain an equation in $z = (16kk')^{\frac{1}{3}}$ corresponding to $\omega = \sqrt{-127}$, $\sqrt{-95}$ and $\sqrt{-31}$.

This gives us sufficient number of equations to determine all the co-efficients. A few co-efficients can be easily obtained by means of the "q" formulae. I have obtained the following :—

$a_1 = -136$, $a_2 = 9920$, $a_3 = -129792$, $a_4 = 16^3.158$

$a_5 = -16^4.21$, $a_6 = 16^5$, $b_1 = 4220$, $b_2 = -3392$

$b_3 = -236544$, $b_4 = 64.15808$, $b_5 = -16^4.17$

$c_1 = -12531$, $c_2 = 179008$, $c_3 = -797696$, $c_4 = 1249280$,

$c_5 = -524288$, $d_1 = -22222$, $d_2 = 141520$,

$d_3=-16.13872$, $d_4=16^3.17$, $e_1=10492$, $e_2=-31984$,

$e_3=-133120$, $e_4=16^3.77$, $e'_1=16604$, $e'_2=-36848$,

$f_1=-2801$, $f_2=54336$, $f_3=-102656$, $f_1'=-7037$,

$f_2'=14304$, $g_1=-23448$, $g_2=16.4571$, $g_3=-56576$,

$g_1'=150$, $h_1=900$.

8. *Modular Equation of the 151st order.*

Proceeding exactly as in the preceding Art., we get the modular equation

$$P^{19}+R(a_1P^{16}+a_2P^{14}Q+a_3P^{12}Q^2+a_4P^{10}Q^3+a_5P^8Q^4+a_6P^6Q^5$$
$$+a_7P^4Q^6+a_8P^2Q^7+a_9Q^8)+R^2(b_1P^{13}+b_2P^{11}Q+b_3P^9Q^2+b_4P^7Q^3$$
$$+c_5P^5Q^4+c_6P^3Q^5+c_7PQ^6)+R^3(c_1P^{10}+c_2P^8Q+c_3P^6Q^2$$
$$+c_4P^4Q^3+c_5P^2Q^4+c_6Q^5)+R^4(d_1P^7+d_2P^5Q+d_3P^3Q^2+d_4PQ^3)$$
$$+R^5(e_1P^4+e_2P^2Q+e_3Q^2)+R^6(f_1P)=0\,;$$

where

$a_1=-4612$, $a_2=4.23328$, $a_3=-16.44596$, $a_4=64.42608$

$a_5=-256\ 22576$, $a_6=1024.6872$, $a_7=-1188.4096$,

$a_8=108.16384$, $a_9=-256.1024$, $b_1=-16.458$,

$b_2=-16.2744$, $b_3=64.20924$, $b_4=-256.23372$.

$b_5=1024.9536$, $b_6=-1452.4096$, $b_7=72.16384$,

$c_1=64.4154$, $c_2=-64.66712$, $c_3=64.267280$,

$c_4=-128.\ 194432$, $c_5=256.46848$, $c_6=-1024.1664$,

$d_1=256.5073$, $d_2=-256.23432$, $d_3=64.136512$,

$d_4=-256.10816$, $e_1=-435200$, $e_2=843776$,

$e_3=-128.6144$, $f_1=32768$.

4. I have also calculated the following complex multiplication moduli.

Let $x=\sqrt[4]{16kk'}$.

$\Delta=135.$

$$(2x^3-3x^2+6x+11)^2-5(3x^2-5)^2=0.$$

$\Delta=147.$

$$(x^3-3x^2+31x-1)^2-21.36x^2=0.$$

Let $s=\left(\dfrac{kk'}{4}\right)^{\frac{1}{12}}$

$\Delta=363.$

$$16s^{12}+32\sqrt[3]{2}\,s^{11}-176s^9+176\sqrt[3]{4}\,s^7-176\sqrt[3]{2}\,s^6+88s^3$$
$$-8\sqrt[3]{4}\,s+1=0.$$

The last two complex multiplication moduli were obtained by putting $\lambda\lambda'=\frac{1}{4}$ in the modular equations of the 7th and 11th orders respectively.

My best thanks are due to Dr. Ganesh Prasad who kindly suggested the problem to me and took great interest in the preparation of the paper.

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NOTICE

The twentieth anniversary of the foundation of the Calcutta Mathematical Society.

At its last annual meeting, the Calcutta Mathematical Society resolved to celebrate the twentieth year of its foundation by bringing out in Volume XX of its *Bulletin*, to be also called the Commemoration Volume, a collection of original papers, by eminent mathematicians all over the world, on Pure Mathematics, Applied Mathematics, Astronomy and the History of Mathematics and Astronomy. The Society also resolved that the President, Dr. Ganesh Prasad, Hardinge Professor of Higher Mathematics, be requested to invite contributions to the Commemoration Volume. It is gratifying to note that the invitations sent out by Professor Ganesh Prasad have met with very encouraging response. Most of the mathematicians, invited to send contributions, have expressed their appreciation of the honour done to them by the invitation, and many of them have either already sent their contributions or promised to send them in the near future. Papers have been received from Sir Frank Dyson (Astronomer Royal of England) and Professors Sir Joseph Larmor (Cambridge), Horace Lamb (Cambridge), A. R. Forsyth (London), Ludwig Bieberbach (Berlin), Leonida Tonelli (Bologna), Leopold Fejér (Budapest), R. Fueter (Zürich), F. Riesz (Szeged, Hungary), W. Sierpinski (Warsaw), Hans Hahn (Vienna), E. T. Whittaker (Edinburgh), Niels Nielsen (Copenhagen), D. E. Smith (New York) and T. Hayashi (Sendai, Japan). About fifteen other papers are expected to reach the President in the next few weeks. The Commemoration Volume will be issued in four parts, each consisting of about 80 pages. The authorities of the Calcutta University Press have kindly consented to make special arrangements for printing the volume, and it is hoped that the first part will be out in the first week of October next.

The 4th August, 1928.

Nripendranath Sen, D.Sc.,

Honorary Secretary.

1

The source of the Indian solution of the so-called Pellian equation.

BY

Sarada Kanta Ganguli

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The equation $Nx^2+1=y^2$ has been erroneously named after Pell who had no connection with it. So far as our present knowledge goes, the first Indian who attempted to find integral solutions of the equation and was partially successful is Brahmagupta, as we shall see later on. His method * is as follows:

If k and k' are found so that

$$Na^2+k=b^2 \qquad \dots (1)$$

and

$$Na'^2+k'=b'^2, \qquad \dots (2)$$

then

$$N(ab' \pm a'b)^2+kk'=(bb' \pm Naa')^2. \qquad \dots (3)$$

This proposition will hereafter be referred to as Brahmagupta's Lemma.

* See Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhāskara*, p. 363. This book will hereafter be referred to as Colebrooke's *Translation* or simply as Colebrooke.

Also see Heath, *Diophantus of Alexandria*, 2nd ed., p. 282; and P. C. Sen Gupta. *Bulletin of the Calcutta Mathematical Society*, Vol. X, p. 74.

In Brahmagupta's Lemma stated below N is called the *multiplier*, k the *kṣepa* which may be positive or negative; a and b are respectively called by Brahmagupta the first root and the last root of the equation $Nx^2+k=y^2$. Colebrooke translates the *kṣepa* as *additive* or *subtractive* according as it is positive or negative. After some commentator (most probably Pṛthudaka Svāmī) Colebrooke usually calls b and a the "greatest" and the "least" root respectively for which he uses the abbreviations G and L. But this is not always true when the *kṣepa* is negative. Hence, confusion is likely to arise in some cases if one depends on Colebrooke's *Translation* only.

It, therefore, follows that

$$N\left(\frac{ab'+a'b}{\sqrt{kk'}}\right)^2+1=\left(\frac{bb'+Naa'}{\sqrt{kk'}}\right)^2 \qquad \dots \quad (4)$$

If kk' be a perfect square,

$$\frac{ab'+a'b}{\sqrt{kk'}} \text{ and } \pm\frac{bb'+Naa'}{\sqrt{kk'}}$$

are rational roots of the equation $Nx^2+1=y^2$.

If kk' be not a perfect square, the relation (2) is to be replaced by the relation (1), in which case the roots become $\frac{ab+ab}{k}$ and $\pm\frac{b^2+Na^2}{k}$. The lower sign gives the roots $(0, \pm 1)$ which must be left out of consideration. Hence, we retain the upper sign only. Relations (3) and (4) thus become

$$N\,(2ab)^2+k^2=(Na^2+b^2)^2, \qquad \dots \quad (5)$$

and $$N\left(\frac{2ab}{k}\right)^2+1=\left(\frac{Na^2+b^2}{k}\right)^2 \qquad \dots \quad (6)$$

Thus Brahmagupta is able to find as many rational solutions of the Pellian equation as he likes.

In case k has any one of the values ± 1, ± 2, ± 4, he can solve the equation in integers by repeated application of his Lemma, * and find as many solutions as may be required.

It is Bhāskara who has first given a method of solving the equation in integers for all values of k.† It consists in repeated application of a corollary from Brahmagupta's Lemma until a solution of the equation $Nx^2+k=y^2$ (where $k=\pm 1$, ± 2, or ± 4) is found.

The corollary is as follows:

* See Colebrooke's *Translation*, pp. 363-366.

† Kaye writes (*Journal of the Royal Asiatic Society*, 1910, p. 755): "Bhāskara gives some alternative methods for the solution of the Pellian equation, but in no essential does he improve on Brahmagupta." This statement shows that Kaye has no idea of the respective contributions of Brahmagupta and Bhāskara to the solution of the Pellian equation in integers. Yet, he writes authoritatively on the Hindu achievements in this matter.

If $Na^2+k=b^2$,

then $$N\left(\frac{aa+b}{k}\right)^2+\frac{a^2-N}{k}=\left(\frac{Na+ba}{k}\right)^2 \quad \dots \quad (7)$$

This corollary will hereafter be referred to as Bhāskara's Corollary.

Here it is necessary that an integral value of a must be found so that $\frac{aa+b}{k}$ may be an integer and a^2-N may be the least. A method of finding such a value has been given by Āryabhaṭa and Brahmagupta.

For the solution of the so-called Pellian equation in integers we are thus indebted to the three Indian mathematicians, Āryabhaṭa, Brahmagupta, and Bhāskara.

In a paper published in the *Bulletin of the Calcutta Mathematical Society*, Vol. X, pp. 73-80, P. C. Sengupta has ably shown that, although Bhāskara's Corollary could be derived from foreign sources in several ways by comparatively laborious processes, Bhāskara obtained it from Brahmagupta's Lemma from which it is an immediate deduction.* But he has not met certain remarks of G. R. Kaye, P. Tannery, and T. L. Heath, which create some doubt as to the indigenous origin of the Indian solution of the Pellian equation and suggest the probability of its ultimate Greek origin. In this paper it is proposed to deal with these remarks.

Kaye writes †: "It is almost certain that Brahmagupta himself did not evolve the solution of the Pellian equation that he gives. There is, indeed, no obvious reason for the inclusion of the topic in Brahmagupta's works, and there are indications that he did not himself comprehend it entirely." He further adds ‡: "Diophantus (A. D. 360) some three centuries earlier had written a treatise on indeterminate

* Bhāskara's Corollary was deduced thus:

$$Na^2+k=b^2$$

$$N.\,1^2+a^2-N=a^2 \text{ (identically)},$$

$\therefore$ by Brahmagupta's Lemma,

$$N(aa+b)^2+k(a^2-N)=(Na+ba)^2$$

$$\therefore \quad N\left(\frac{aa+b}{k}\right)^2+\frac{a^2-N}{k}=\left(\frac{Na+ba}{k}\right)^2.$$

† *East and West* (Simla), July 1918, p. 677.

‡ *Ibid*, p. 677.

equations; some of his books and all those of his successor Hypatia (A.D. 415) are lost; Brahmagupta's treatment of the subject differs in detail from Diophantus; Brahmagupta's examples are all astronomical and Brahmagupta's astronomy is essentially Greek astronomy." In a foot-note he states *: "The known works of Diophantus do not refer to the equation $ax^2+1=y^2$ but a solution of $ax^2-b=y^2$ is given (Lemma to VI. 15)." He thus concludes that Brahmagupta's solution of the Pellian equation is of Greek origin.

Brahmagupta's work *Brahma-sphuṭa-siddhānta* contains a treatment of indeterminate equations (both simple and quadratic), knowledge of which is, according to Brahmagupta (Chapter on *Kuṭṭaka*, verse 2), essential to an astronomer. Brahmagupta has given astronomical problems whose solutions depend on indeterminate equations of the first and second degrees. The reason for the inclusion of the topic is, therefore, obvious. Did the Greeks ever trouble themselves with astronomical problems leading to indeterminate equations? It has yet to be proved that Brahmagupta's astronomy is entirely of Greek origin. To establish Brahmagupta's sources Kaye writes †: "Brahmagupta refers to the Romaka, Paulisa, and other Yavana (Greek) Siddhāntas and says, 'Although the Siddhāntas are many, they differ only in words, not in the subject-matter.' He disclaims originality and definitely indicates the sources of his information." Kaye has not given the reference. I presume that he refers to verse 3 of Chapter XXIV of *Brahma-sphuṭa-siddhānta*. If my surmise is correct, Kaye has omitted to mention the names of three Hindu Siddhāntas ‡ which also occur in the verse and does not seem to have understood it correctly. To understand the verse properly it should be read with verses 2 and 4. These three verses state that Sūrya, Indu, Paulisa, Romaka, Vaśiṣṭha, and Yavana Siddhāntas are similar in some respects and different in others; and they lay down a test for ascertaining the correct *siddhāntas*. These verses cannot be taken to mean that Brahmagupta disclaims originality and indicates Paulisa, Romaka, and Yavana Siddhāntas as his only sources. That Brahmagupta had no regard for Romaka Siddhānta may be seen from his Siddhānta, Chapter XI, verse 50, where he describes Romaka Siddhānta as a heap of ashes made into a *kanthā*, § as it were (*rakṣoocaya-romakaḥ kṛtaḥ kanthā*). In

* *Ibid*, p. 677.

† *Ibid*, p. 679.

‡ See below.

§ It is the name of a kind of Indian quilt made of rags and used by poor people. Here the term is used contemptuously.

many cases (*e.g.*, XI, 46, 47, 48) he finds fault also with Lāṭa who, according to Varāhamihira (Pañca-siddhāntikā, I, 3), explained the Romaka and Paulisa Siddhāntas. On the other hand, he speaks highly of his own Siddhānta and points out its superiority over all other siddhāntas (I. 32, 62; II. 31 ; V. 21; X. 62, 69 ; XI. 61 ; etc., etc.)

Even if it be assumed, for the sake of argument only, that Brahmagupta's astronomy is essentially Greek astronomy, how does it prove that his solution of the Pellian equation was derived from the Greeks who, as we shall see later on, could not solve it ? The fact that the illustrative examples are astronomical, only shows that astronomical problems led to the independent evolution of the solution of indeterminate equations in India.*

In *Brahma-sphuṭa-siddhānta* we have a piece of evidence which points strongly to Brahmagupta as the author of his rules for the solution of equations of the type $ax^2+b=y^2$. The illustrative examples, without exception, † end with the words '*kurvannāvatsarād-gaṇakaḥ*' which mean that a person who can solve each of these examples within a year is a mathematician. Such a statement can proceed only from a mathematician who honestly believes that the examples are all very difficult, that rules for their solution did not exist previously, and that he is the first person to give these rules.

Let us now consider the alleged indications that Brahmagupta did not entirely comprehend his solution of the Pellian equation. These indications have elsewhere ‡ been stated by Kaye while trying to establish a foreign origin of the solution. Kaye there writes : " A close examination of Brahmagupta's rules and examples establishes beyond all doubt that he was not their discoverer. He does not understand all the rules he gives. Some rules are followed by inappropriate examples (Colebrooke, Algebra, p. 366, §72). In one case he partially solves an example, and says : 'The meaning of the rest will be shown further on' (*ibid*, §70). The example is solved further on, but the previous working is not utilized (*ibid*, §77). Another rule and accompanying example are regular, but Brahmagupta gives a second similar example with a change of sign which the rule does not account for ; and while

* See Robert Steele's *The Earliest Arithmetics in English*, Introduction, p. v, where the author says that " the study of astronomy necessitated, from its earliest days as a science, considerable skill and accuracy in computation."

† Colebrooke has omitted the words 'within a year' in Question 29, Art. 72, p. 366, of his translation of Brahmagupta.

‡ *Journal of the Royal Asiatic Society* (1910), p. 755.

correctly explaining this second example, refers to the incompleteness of the text and criticizes the rule given (*ibid*, §73). In another case he finds fault with the rule and says: 'With the exception of a selected unknown put arbitrary values for the rest.........thus the solution is effected without an equation of the second degree. What occasion is there for it?' (*Ibid*, §64.)" *

This extract is made up of a wrong conclusion and a number of incorrect statements made to support it. Let us examine these statements, one by one.

The first statement contains the charge that "some rules are followed by inappropriate examples." The reader may notice that Kaye uses the word 'rules' in the plural number while he can cite only one instance. Even this instance does not support the statement.

The so-called 'inappropriate' examples noticed only by Kaye lead to the indeterminate equations

$$(i)\quad 13x^2+300=y^2, \qquad (ii)\quad 13x^2-27=y^2.$$

These examples immediately † follow the rule for deducing the integral roots of the equation

$$Nx^2+1=y^2$$

from those of the equation

$$Nx^2-4=y^2.$$

To one who is not acquainted with Brahmagupta's rules on this topic there cannot seem to exist any relation between the above rule and the examples set in illustration thereof. To find the roots (10, 40) of

* It is very doubtful whether Brahmagupta included solutions of examples in his works. From the works of his predecessor Āryabhaṭa and his successor Mahāvira it appears that it was not the custom in those days to include solutions of examples in the original works of Indian mathematicians. Examples used to be explained orally and their solutions would be left for the commentators to record. Solutions of examples do not form a part of Brahmagupta's *Brahma-sphuṭa-siddhānta* as edited by Sudhākar Dvivedi who has compiled his edition from three manuscripts different from the one used by Colebrooke in preparing his *Translation* (see Dvivedi's edition, p. 3). Nevertheless, let us assume, for the sake of argument only, that the solutions of examples as given in Colebrooke's *Translation*, are in the words of Brahmagupta himself.

† See Colebrooke's *Translation*, pp. 365 and 366, Arts. 71 and 72. In Sudhākar Dvivedi's edition the above examples occur among the miscellaneous examples following the rules on this particular topic.

equation (i) and the roots (6, 21) of equation (ii) the above rule has not to be applied. But Brahmagupta's object is not to find only a single solution of such an equation but an unlimited number of integral roots. A previous rule of his states that, when one solution of the equation $Nx^2+P=y^2$ has been found, an infinite number of solutions can be obtained by associating with it the equation $Nx^2+1=y^2$ and then applying his Lemma. It is in thus finding innumerable integral solutions of equations (i) and (ii) that the above rule has to be applied. To find other solutions of these equations use should be made of the roots of the equation $13x^2+1=y^2$. These roots are given by the rule immediately preceding the above examples. For, the roots (1, 3) of the equation $13x^2-4=y^2$ are obvious.

It will thus be seen that Kaye's charge under consideration is without any basis whatsoever. It is strange that the commentator's solution of the above examples, as given in Colebrooke's *Translation*, did not open Kaye's eyes to the utter hollowness of his allegation. It is true that the solution is incomplete.* But the language in which it is expressed, if not its subject-matter, is sufficient to point out the intimate connection that certainly exists between the above examples and the preceding rule.

The second statement is as follows :—

"In one case he partially solves an example, and says: 'The meaning of the rest will be shown further on.' The example is solved further on, but the previous working is not utilized."

On a reference to Colebrooke's *Translation* (pp. 365 and 368) the facts will be found to be as follows :—

Two examples lead to the indeterminate equations

$$(i)\quad 3x^2+900=y^2,$$

$$(ii)\quad 3x^2-800=y^2.$$

These examples have been first given to illustrate the rule (Colebrooke, §69, Rule 42, p. 365) for deducing a solution in integers of the equation

$$Nx^2+1=y^2$$

* Hence the commentator says: "The purport of the rest of the question will be shown further on."

from an integral solution of the equation

$$Nx^2+4=y^2.$$

For, to find an unlimited number of solutions of either of the above two equations by means of Brahmagupta's Lemma use will have to be made of the integral roots of the equation

$$3x^2+1=y^2,$$

and these roots can also be found by the above rule since 2 and 4 are the obvious roots of the equation

$$3x^2+4=y^2.$$

Here the roots of the equation $3x^2+1=y^2$ have been found by the above rule: and the complete solution of the examples is not given here, since it is proposed to be worked out fully hereafter in illustration of another rule. Hence, we have the remark: "The meaning of the rest will be shown further on."

The other rule (Colebrooke, p. 368, Rule 45 completed) which the above examples illustrate states that the roots of the equation

$$Nx^2+k=y^2,$$

when multiplied by a, are the roots of the equation

$$Nx^2+ka^2=y^2.$$

Here the equations (*i*) and (*ii*) are solved by this rule as follows:

(*i*) $$3x^2+900=y^2$$

~~or,~~ $$3x^2+1.30^2=y^2.$$

Now, 1 and 2 are the obvious roots of

$$3x^2+1=y^2.$$

$\therefore$ 30 and 60 are a pair of roots of equation (*i*).

(*ii*) $$3x^2-800=y^2$$

or, $$3x^2-2.20^2=y^2$$

Obviously, 1 and 1 are the roots of

$$3x^2-2=y^2.$$

$\therefore$ 20 and 20 are a pair of roots of equation (*ii*).

It is true that the roots of the equation $3x^2+1=y^2$, previously obtained by applying Rule 42 (Colebrooke, p. 365) have not been utilised here. Will any one use them in the present case, seeing that the roots 1 and 2 are too obvious? If these obvious roots do not occur to a person, he may utilise the roots previously obtained but it is advisable for him to give up the study of indeterminate equations.

" The meaning of the rest " " to be shown further on" is here made clear by fully working out the examples and showing how the roots of equations of the form $Nx^2+1=y^2$, obtained by inspection or by the application of Rule 42 or of any other suitable rule, should be utilised to find other pairs of roots. Let me quote from the solution attributed to Brahmagupta.

"And further, by composition of the roots so found for the given additive, with roots serving for additive unity, other roots are derived for the same additive. For instance,

L 30 G 60 A 900

L 1 G 2 A 1

By the preceding rules (§39-40 and 41) other 'least' and 'greatest' roots are here found, 120 and 210. So in all similar cases."

It will be seen from the above that the question of utilising the previous working cannot arise here and that no mathematician, even in the present case, would ever think of rejecting obvious roots in favour of roots which are not obvious.

The third statement runs thus :—

" Another rule and accompanying example are regular, but Brahmagupta gives a second similar example with a change of sign which the rule does not account for ; and while correctly explaining this second example, refers to the incompleteness of the text and criticizes the rule given (§73)."

Colebrooke translates the rule in question as follows :—

" If a square be the multiplier, the additive (or subtractive) divided by any (assumed) number, and having it added and subtracted, and being then (in both instances) halved ; the first is a 'last' root ; and the last, divided by the square root of the multiplier, is a 'first.' "

This rule is illustrated by two examples which lead respectively to the equations

$$(i)\quad 4x^2+65=y^2,$$

$$(ii)\quad 4x^2-60=y^2.$$

Kaye thinks that the first example is regular and that the negative sign in equation (*ii*) has made the second example inappropriate. For, in his opinion, the rule cannot " account for " the negative sign.

Colebrooke has given an almost word-for-word translation of the text of Brahmagupta's rule in question. He ought to have left the word *kṣepa* untranslated without rendering it as 'additive.' The insertion of the words "or subtractive" within brackets does not mend matters fully. For, *kṣepa* means a quantity which is added either positively or negatively and, therefore, involves a sign. This interpretation should not be regarded as being in advance of Brahmagupta's time. For, in a previous chapter he has dealt with negative quantities. Also, the terms *prakṣepa* and *kṣepa* in the expression "*prakṣepaḥ kṣepavadha-tulyaḥ*," which occurs in the text of Brahmagupta's Lemma, appear, from the context, to be indefinite as regards their sign.

For the present let us overlook this defect in Colebrooke's translation.

One who is acquainted with the nature of Sanskrit *sutras* or texts knows very well that to secure brevity they are intended by their framers to imply much more than a word-for-word translation could possibly convey. The full implication is to be found by the well-known principle of *upalakṣmaṇa*. Hence, although the Sanskrit text of a rule may state one case, it is taken to include, by implication, other similar cases (suggested by it) for which separate or additional texts are generally never thought necessary.* The text of Brahmagupta's rule, as translated by Colebrooke, states the case for positive *kṣepa* and, as usual, includes the case for negative *kṣepa* as well. Hence, we have the

* Compare Brahmagupta's rule (§ 45) which Colebrooke translates (p 367) as follows :—

"If the multiplier be (exactly) divided by a square, the first root is (to be) divided by the square-root of the divisor."

The commentator states the full meaning of this rule thus :—

" If the multiplier can be abridged by a square, then reducing to its least term, let roots be found as before. But the first root so found being divided by the square-root of the abridging divisor, is the ' least ' root. The 'greatest' root remains the same."

" But if the coefficient be multiplied by a square quantity, it of course follows that the first root, multiplied by the square-root of that square, is the ' least ' root." (Colebrooke, p. 367, foot-note.)

Also compare Āryabhaṭa's rule (considered below) for finding a number which leaves given remainders on being divided by two given divisors (p. 174 below),

following observations in course of the solution of the second example intended to illustrate the rule :

" 'Additive' in the rule (§44) is indefinite and intends subtractive also."

" The text expresses ' the first is a last root ' (§44) : but that is a part only of the rule."

By these observations the author of the solution means that, as usual, the text does not expressly state the whole rule but a part only of it and necessarily suggests the remaining part. But Kaye, in his ignorance of Sanskrit and of the nature of Sanskrit texts of rules, takes these observations to imply that " Brahmagupta refers to the incompleteness of the text and criticizes the rule given." What adverse criticism has been made ? Will Mr. Kaye or any of my readers kindly point it out to me ?

Although the text of the rule in question, like the Sanskrit texts of many rules, may be incomplete, the whole rule for which the text stands covers the cases of both the positive and the negative *kṣepa*. Hence, the rule does (although its text may not) account for the negative sign in the equation resulting from the second example.

The above lengthy explanation in support of Brahmagupta has been necessitated partly by Colebrooke's translation of the rule and partly by he assumption that Brahmagupta's original work contains solutions of texamples.

If we do not make this assumption, the truth of which we have reasons to doubt, and if we go by the rule as given in Sanskrit by Brahmagupta and not as translated into English by Colebrooke, we at once see that, if Brahmagupta had omitted the second example taken exception to by Kaye, he could have been justly accused of having failed to illustrate the rule completely.

As has already been said, the *kṣepa* in the rule may be positive or negative and has not been qualified in any way. Since the rule in question follows the chapter on negative quantities and since the sign of *kṣepa* and *prakṣepa* in Brahmagupta's Lemma is indefinite (as stated above), one is not justified in taking the *kṣepa* as positive only. Hence, the rule covers the case for negative *kṣepa* also, as shown below.

In modern language the rule may be stated thus :

If the equation be $a^2x^2+b=y^2$,

$$\frac{1}{2}\left(\frac{b}{k}+k\right) \text{ and } \frac{1}{2}\left(\frac{b}{k}-k\right)\div a$$

are respectively the values of y and x, k being any assumed integer (having the same sign as b).

The above are the expressions for the roots whether the sign of b is positive or negative. As these roots are obtained by substituting $ax+k$ for y, k must evidently have the same sign as b.

The reader will now be able to judge for himself whether this rule does or does not account for the negative sign in equation (*ii*).

Let us now consider the fourth or the last statement:

In Art. 64 (Colebrooke's *Translation*, p. 362) Brahmagupta finds fault with a rule included in his work.* This fact has been utilised by Kaye in his favour.† The facts are as follows:

In the section on Bhāvita-bīja which immediately precedes the section on the Pellian equation Brahmagupta first states a rule for the solution of equations of the form

$$mxy=ax+by+c$$

The commentator holds that this rule is not Brahmagupta's own but has been given by others. Then Brahmagupta gives a rule (or strictly speaking, a method) of his own, points out its superiority over the preceding rule and remarks "What, then, is the occasion for it (*viz.*, the first rule)?" Such criticism cannot prove that Brahmagupta does not understand the rule he criticises.‡ Hence, the statement under con-

* This statement and the following discussion are based on the assumption that the commentator's view is correct. But I should like to differ from him. The fact that the first rule (§61) is followed by an example which challenges a person to solve it within a year goes to show that Brahmagupta is the author of this rule also. He then gives an alternative method of his own and points out its superiority over the first method as given by the first rule, without passing any censure on the first method. It is the duty of a mathematician to show the different methods of attacking a problem and discuss their relative merits and demerits. It cannot amount to passing a censure on an inferior method. Since Brahmagupta regards the first method as inferior, the commentator thinks that the former censures the first method and, therefore, cannot be its author. If my view be correct, then also it does not follow that Brahmagupta does not understand the rule in question.

† To prove Brāhmagupta's failure (as alleged by Kaye) to understand not only his rule for the solution of the Pellian equation but also Āryabhaṭa's rule for the solution of indeterminate equations of the first degree as will be shown hereafter.

‡ Kaye could find far better reasons to hold that Diophantus does not understand all his methods. For, he cannot solve, he says, the equations $52x^2+12=y^2$ (III. 10) and $266x^2-10=y^2$ (III. 11). But $x=1$ is an obvious solution in either case and other solutions can be found by his Lemma to VI. 15. Still we regard him as a genius.

sideration—as it occurs in the above extract—does not go against Brahmagupta. But, elsewhere* Kaye has expressed it in a way which cannot but discredit Brahmagupta. Kaye writes: "On one occasion he (*i.e.*, Brahmagupta) gives a correct rule (XVII, iv, 61)† but immediately discards it saying, 'What occasion is there for it? .... with the exception of one unknown put arbitrary values for the rest.',‡ and the commentator remarks: 'The author here delivers his own (incorrect) method with a censure on the other (correct method).'." Here Kaye unaccountably puts into the mouth of the commentator a remark never made by the latter. The commentator remarks (Colebrooke, p. 362, F. N. 1): "Having thus set forth the (solution of a) factum according to the doctrine of others, the author now delivers his own method with a censure on the other." Here Kaye misleads his readers by inserting three words within brackets without any justification whatsoever. The commentator does not consider Brahmagupta's method to be incorrect. The fact that Kaye characterises this method as incorrect shows that he has utterly failed to understand it. Brahmagupta means to say that there is no occasion for the first rule since the second rule is much simpler than it. He does not say that the first rule is incorrect. ThatBrahmagupta is right will appear from a comparison of the two methods.

Brahmagupta's method, i.e., *the second method*

$$mxy=ax+by+c$$

Putting any assumed value k for x, we have

$$kmy=ak+by+c,$$

whence y can be easily found.

This method is applicable, as stated by Brahmagupta, to such an equation with any number of unknown quantities.

* *Journal of the Asiatic Society of Bengal*, (March, 1908), p. 138.

† This reference is wrong. The correct reference should be "(Colebrooke's *Algebra, etc.* Brahmagupta, XVIII, vi, 61)."

‡ The reader's attention is drawn to the alterations made by Kaye to get this quotation out of the quotation given on page 156 (lines 4-6). These alterations have become necessary. For, to create a presumption in favour of his pet theory of the Greek origin of Āryabhaṭa's rule for the solution of indeterminate equations of the first degree Kaye has used this altered quotation which would mislead his readers into thinking that Brahmagupta also "was not quite master of this part of his subject."

The other method (i.e., *the first method*) :

$$mxy = ax + by + c.$$

Multiply m by c and a by b, and add the products. The result is $cm+ab$. Divide the sum by any assumed number, say p, and let q be the quotient. Of the two pairs of numbers (p, q) and (a, b), the greater number of each pair should be added to the smaller number of the other pair.* Suppose that we thus get $a+p$ and $b+q$.

Then $x=(b+q)/m$ and $y=(a+p)/m$.

The *rationale* of this rule is as follows :

From the given equation we easily obtain

$$m^2xy - amx - bmy + ab = cm + ab,$$

or $$(mx-b)(my-a) = cm + ab \equiv pq \quad \text{(suppose)}.$$

Then, either $x=(b+p)/m$ and $y=(a+q)/m$,

or $x=(b+q)/m$ and $y=(a+p)/m$.

This method is not applicable to an equation of the type

$$mxyz = ax + by + cz + d.\dagger$$

The foregoing examination of the statements made by Kaye to discredit Brahmagupta forces me to the conclusion that he has taken upon himself the serious responsibility of criticising Brahmagupta and denouncing him to the world without ever caring to understand him or even approaching him with an unprejudiced mind.

Let us now discuss the possibility of Brahmagupta's indebtedness to Diophantus for his solution of the Pellian equation. It is true that the known works of Diophantus do not refer to the general equation

* This restriction is not necessary, as shown below.

† These cases show that Kaye is quite unfit to undertake a detailed examination of Indian mathematical works. He, therefore, always gives a 'brief' examination (*J. R. A. S.*, 1910, p. 750) or 'a part only' (*East and West*, July, 1918, p. 679) of the available evidence to prove the alleged foreign origin of Indian mathematical works. He tries to thrust his anti-Indian views on his readers by pointing out some accidental superficial similarities which must exist between two independent systems, as admitted by him. He says (*J. L. A. S.*, 1910, p. 750) ; " While mathematical systems of independent growth will naturally have many points of similarity, yet differences are certain to occur."

$ax^2+1=y^2$. But they contain a solution of at least one particular case of this general equation. Diophantus solves equations of the form $ax^2+c^2=y^2$ by putting $y=kx\pm c$ where k^2 is the highest square number under a. He solves three such equations as follows :

(*i*)	$72m^2+81=(8m+9)^2$	IV. 31
(*ii*)	$26x^2+1=(5x+1)^2$	V. 9
(*iii*)	$65m^2+25=(8m+5)^2$	VI. 5

To reduce $ax^2+b=y^2$ to a simple equation either ax^2 or b must be removed. This can be easily done when the term to be removed is a perfect square. This would occur to any student of mathematics far inferior in intellect to Diophantus, Āryabhaṭa and Brahmagupta. So we have no reason to be charmed with Diophantus' method of rationally solving equations of the form $ax^2+b=y^2$ (where a or b is a perfect square) by substituting $(\sqrt{a}x+k)$ or $(kx+\sqrt{b})$ for y. Brahmagupta also could do so if his object were to obtain rational solutions only.* The fact that Brahmagupta does not use this natural method and shows how an unlimited number of integral solutions can be found in particular cases with the help of his Lemma leads us to believe that he starts with the object of finding integral solutions in all cases although he does not succeed completely.

Kaye writes that Diophantus has given a solution of the equation $ax^2-b=y^2$ (Lemma to VI. 15). This is not true. The lemma states that, *if one value of x satisfying the equation is known*, a greater value of x satisfying the equation can be found. Diophantus admits (VI. 14) that he cannot solve the equation " $15m^2-36=$ a square " because 15 is not the sum of two squares.

Again, it is not true that Brahmagupta's t[redacted]ent of indeterminate analysis differs only in detail from Diophantus '. This will be clear from a comparison of the methods employed by Diophantus and Brahmagupta to solve the same equation. Let us take the equation $26x^2+1=y^2$ from Diophantus.

Diophantus' method :

$$\text{Let } y=1+kx.$$

* Brahmagupta actually solves the equation $a^2x^2+b=y^2$ by assuming $y=ax+k$ (Colebrooke, p. 366, 78). For, owing to the equation $a^2x^2+1=y^2$ being incapable of solution in integers, it is not possible to find an unlimited number of integral solutions of the equation $a^2x^2+b=y^2$.

Then $26x^2+1=(1+kx)^2$,

whence $x=\dfrac{2k}{26-k^2}$.

Brahmagupta's method :

Since $26.1^2-1=5^2$ and $1=(-1)\times(-1)$,

the roots of the equation $26x^2+1=y^2$

are, by Brahmagupta's Lemma, 2.1.5 and 26.1^2+5^2, *i.e.*, 10 and 51.

The next pair of roots are, by the Lemma, 2.10.51 and 26.10^2+51^2 *i.e.*, 1020 and 5201.

The third pair of roots are, by the Lemma, 10.5201+51.1020 and 26.10.1020+51.5201, *i.e.*, 104080 and 530451.

Another pair of roots are, by the Lemma, 2.1020.5201 and 26.1020^2+5201^2, *i.e*, 10610040 and 54100801.

And so on.

Diophantus' method gives one pair of integral roots when $k=5$. It is not easy to see what other values of k also lead to integral solutions.

Tannery is of opinion that Diophantus dealt with the equation $Nx^2+1=y^2$ somewhere in the lost Books of the *Arithmetica* and suggested a method of solution. Heath has shown (*Diophantus*, 2nd ed., p. 280) that " here too we must have ascertained two solutions of the equation, or one solution of it and a solution of an auxiliary equation, before we can apply the method." But is it likely that the equation was solved in integers or dealt with at all in the lost Books of the *Arithmetica* ? Our answer is in the negative for the following reasons :

(1) Diophantus solves the equations

(*i*) $72m^2+81=$ a square,

(*ii*) $26x^2+1=$ a square,

(*iii*) $65m^2+25=$ a square,

respectively in IV. 31, V. 9, and VI. 5 of the *Arithmetica* and employs the same method as explained before. As already observed, this method cannot give many integral roots, not to speak of an infinite number of

them. The solutions of equations (*i*) and (*iii*) follow at once from those of the equations

$$m^2+1=y^2 \quad \text{or } 8m^2+1=y^2,$$

and $$65m^2+1=y^2.$$

The proper place for a special treatment of the equation

$$Nx^2+1=y^2$$

is consequently Book V, if not Book IV, and not any of the lost Books.

(2) If a method of finding integral solutions of the equation in question had been given in any of the lost Books, Diophantus would have very probably referred to it in connection with the solution of the above three equations as he has done in Book V where the references are to the Porisms.

(3) It is not likely that equations of the type $Nx^2+1=y^2$ would have been solved *rationally* only in the known Books if they had been solved in integers in the lost Books.

(4) If Diophantus had been aware of a general solution in integers of the equation $Nx^2+1=y^2$, he would have given more integral solutions than one of each of the equations (*i*), (*ii*), (*iii*).

(5) Heath writes (*Diophantus*, p. 67): "Nor is it likely that indeterminate equations of the first degree were treated of in the lost Books. For, as Nesselmann observes, while with indeterminate quadratic equations the object is to obtain a *rational* result, the whole point in solving indeterminate simple equations is to obtain a result in *integral* numbers. But Diophantus does not exclude fractional solutions, and he has therefore only to see that his results are *positive*, which is of course easy. Indeterminate equations of the first degree would therefore, from Diophantus' point of view, have no particular significance." If indeterminate equations of the first degree were not treated of in the lost Books and if with indeterminate quadratic equations Diophantus' object, so far as it appears from the knowm Books, has always been to obtain a rational result, is it likely that the much harder case of solution of the equation $Nx^2+1=y^2$ in integers was ever dealt with by him?

If we suppose, for the sake of argument only, that Diophantus had a method of solving $Nx^2+1=y^2$ in integers, Brahmagupta's indebtedness to him, as alleged by Kaye and others, is at once disproved. For, Brahmagupta cannot solve the general equation in integers.

Regarding Kaye's suggestion as to the possibility of Brahmagupta's indebtedness to the lost Books of the *Arithmetica* or to the works of Hypatia, I may add that in his *Indian Mathematics*, which seems to have been written to decry Indian mathematicians, Kaye perceived the untenability of the suggestion. There he wrote (p. 15) : "It would be pleasant to conceive that in the Indian works we have some record of the advances made by Hypatia or of the contents of the lost Books of Diophantus—but we are not justified in indulging in more than the mere fancy."

Let us now see if the Indian solution of the Pellian equation can be traced to Archimedes. The Pythagoreans discovered a general method of solution of $2x^2-y^2=\pm 1$ to obtain successive approximations to $\sqrt{2}$.* Archimedes gives the fractions 265/153 and 1351/780 as being approximately equal to $\sqrt{3}$.† It has, therefore, been suggested that Archimedes was acquainted with a method of solution of the equations

$$x^2-3y^2=1,$$

$$x^2-3y^2=-2.$$

The famous Cattle-Problem attributed to him requires the solution in positive integers of the equation

$$x^2-4729494y^2=1.$$

But Heath writes : "There is this difference between this equation and the simpler ones above that, while the first solutions of the latter can be found by trial, the simplest solution of this equation cannot, so that some general method, *e.g.*, that of continued fractions, is necessary to find even the least solution in integers. Whether Archimedes was actually able to solve this particular equation is a question on which there is difference of opinion ; Tannery thought it not impossible, but, as the smallest values of x, y satisfying the equation have 46 and 41 digits respectively, we may, with Günther, feel doubt on the subject. There is, however, nothing impossible in the supposition that Archimedes was in possession of a general method of solving such equations where the numbers involved were not too great for manipulation in the Greek numeral notation." ‡ If Archimedes was really in possession of a method of solving the Pellian equation in integers, it is certain that

* Heath, *Diophantus*, pp. 117, 118, 278.

† *Ibid*, p. 278.

‡ *Ibid*, p. 279.

neither Diophantus nor Brahmagupta was aware of it. As Bhāskara's method has been deduced from Brahmagupta's Lemma, the former's ultimate source cannot be Archimedes.*

Again, the mere fact that the Cattle-Problem leads to an equation of the form $x^2 - ay^2 = 1$ cannot prove that Archimedes could solve such equations in integers. Has he given a solution of the equation? The proposer of a problem may not always be able to solve it. When Fermat states that he has "discovered a truly marvellous proof" of his "great theorem," he is not believed.† But Archimedes and Diophantus must be credited with a method which they do not claim to know. Is it because Indian mathematicians must be deprived of an honour that really belongs to them? ‡

Let me now deal with certain observations made by Heath. He writes (*Diophantus*, p. 281): "Whether this method (*i.e.*, the Indian solution of the Pellian equation) was evolved by the Indians themselves, or was due to Greek influence and inspiration, is disputed.................... The question presumably cannot be finally decided unless by the discovery of fresh documents; but, so far as the other cases of solution of indeterminate equations by the Indians help to suggest a presumption on the subject, they are, I think, rather in favour of the hypothesis of ultimate Greek origin. Thus the solution of the equation $ax - by = c$ given by Aryabhaṭa as well as by Brahmagupta and Bhāskara.........is an easy development from Euclid's method of finding the greatest common measure or proving by that process that two numbers have no common factor (Eucl. VII 1, 2; X. 2, 3), and it would be strange if the Greeks had not taken this step. The Indian solution of the equation $xy = ax + by + c$, by the geometrical form in which it was clothed, suggests Greek origin."

Although I yield to none in my regard for Heath as ~~an~~ impartial historian of mathematics, I am bound to observe that this is one of at least two cases in which he has unconsciously allowed himself to accept the opinion of others without sufficient examination. In the present case he has been misled by Kaye to whom he here refers as his authority.

* P. C. Sen Gupta has shown (*Bulletin*, Cal. Math. Soc., Vol. X, pp. 78-80) that, although Bhāskara's Corollary can be deduced from the Archimedian process of finding the approximate value of $\sqrt{N}$, it follows at once from Brahmagupta's Lemma. The Corollary cannot give the first solution in most cases. To obtain it and the succeeding solutions Brahmagupta's Lemma must be employed.

† Heath, *Diophantus*, p. 145, foot-note.

‡ Such is the object of Kaye's writings which do not seem to have been prompted by a desire to investigate the truth.

The Indian solution of the equation $ax-by=c$ may *now* appear to be an easy development from some of Euclid's methods. But the fact remains that the Greeks could not effect this easy development. After a thing has been discovered it is sometimes very easy to obtain it from other sources by simple steps. Many familiar instances can be cited in support of this statement. Here is one of them. Now we see that the modern notation for the number four thousand five hundred and seventy-six follows at once from the way in which it is expressed in words ; and one might remark that it would be strange if the Greeks had not invented this simple notation. Would it be right to build up a theory of Greek origin of the modern notation on this sort of argument ? The question is not whether Āryabhaṭa's rule can be derived from Greek or some other foreign source but whether it was actually derived in that way.

N. K. Mazumdar has shown that Āryabhaṭa was not indebted to Euclid or other Greek or Alexandrian mathematicians for his rule for the solution of $ax-by=c$ and has pointed out some important mistakes in Kaye's investigations in connection with the rule.* In reply Kaye writes : † " Āryabhaṭa's text itself is by no means unambiguous and my original translation (used by Mr. Mazumdar) does not satisfy me. Conclusions based upon such a text are very liable to contain ambiguities." He then adds : " I still hold that the fundamental process involved in Āryabhaṭa's rule is contained in Euclid (Books *ii* and *vii*) but this does not necessarily mean that the rule was a direct development from Euclid. The point is that nowhere in Indian works do we meet with any process at all that could lead up to the rule." ‡

Kaye's translation is wrong partly because he has interpreted the terms *adhikāgrabhāgahāra* and *ūnāgrabhāgahāra* respectively as ' the greater original divisor ' and 'the lesser original divisor.' The former term means ' the divisor (*bhāgahāra*) which gives the greater (*adhika*) remainder (*agra*)' and the latter 'the divisor which gives the smaller (*ūna*) remainder.' Āryabhaṭa's rule states that the former should be divided by the latter.

* *Bulletin*, Cal. Math. Soc., Vol. III, pp. 11-19.

† *Bulletin*, Cal. Math. Soc., Vol. IV, p. 55.

‡ Kaye further adds : " There is certainly an element of doubt as to the authenticity of the text." A comparison of this text with the original rules in Sanskrit given by Brahmagupta and junior Āryabhaṭa will at once show that the element of doubt is more imaginary than real. The present writer has good reasons to believe that Kaye is ignorant of Sanskrit and depends only on English translations of Indian works.

Mazumdar begins by stating Kaye's wrong translation of Āryabhaṭa's rule for the solution of the general problem which Mazumdar states as follows:

"To find a number n which will leave given remainders when divided by given positive integers; *i.e.*, n is to be found from the set of equations

$$n/A = x + R_1/A, \qquad n/B = y + R_2/B,$$

which leads to the indeterminate equation

$$Ax - By = C."$$

Here C is the difference between the remainders R_1 and R_2 and is, therefore, positive. Hence Mazumdar takes R_2 to be greater than R_1. He also assumes B to be greater than A. Therefore he considers the particular case in which the divisor yielding the greater remainder happens to be greater than the other divisor.* In this case, even Kaye's translation, *viz.*, "*the greater original divisor is divided by the lesser original divisor*," though wrong, gives the first operation correctly. The other wrong part of Kaye's translation, *viz.*, "*An assumed number together with the original difference is thrown in*," has not been used by Mazumdar who has interpreted this part of the rule correctly as shown by his application of it. Hence, Mazumdar's treatment of the particular case considered by him is not based on a wrong interpretation of Āryabhaṭa's rule as applicable to this case. His conclusions, therefore, are not liable to contain ambiguities.

Āryabhaṭa's rule for the solution of the general problem stated above was most probably suggested by simple and natural steps dictated by common sense in particular cases as shown below.

Let us find the number which gives the remainders 9 and 17 when it is divided by 931 and 3015 respectively.

If x and y be the respective quotients, the required number is given by either of the expressions $931x+9$ and $3015y+17$. Hence the problem leads to the equation

$$931x + 9 = 3015y + 17 \qquad \dots \quad (1)$$

* P. C. Sen Gupta has interpreted Āryabhaṭa's rule in question in the Journal of the Department of Letters (Calcutta University), Vol. XVI, pp. 27-30. But he also has not shown why the divisor giving the greater remainder should be divided first by the other divisor. This part of Āryabhaṭa's rule will also be made clear in the following pages.

It can be solved if we can solve

$$931(x-3y)+9=222y+17$$

or, $$931x_1+9=222y+17 \qquad \dots \ (2)$$

where $x_1=x-3y$.

(2) can be solved if we can solv

$$43x_1+9=222\ (y-4x_1)+17$$

or, $$43x_1+9=222y_1+17 \qquad \dots \ (3)$$

where $y_1=y-4x_1$.

(3) can be solved if we can solve

$$43(x_1-5y_1)+9=7y_1+17$$

or, $$43x_2+9=7y_1+17 \qquad \dots \ (4)$$

where $x_2=x_1-5y_1$.

(4) can be solved if we can solve

$$x_2+9=7(y_1-6x_2)+17$$

or, $$x_2+9=7y_2+17 \qquad \dots \ (5)$$

where $y_2=y_1-6x_2$.

Since the coefficient of one of the unknown quantities in equation (5) is unity, we can solve it in integers. On taking $y_2=i$ (any positive integer including zero)

we get

$$x_2=7i+(17-9) \qquad \dots \ (6)$$

Now, from (2) $\quad x=3y+x_1 \qquad \dots \ (7)$

... (3) $\quad y=4x_1+y_1 \qquad \dots \ (8)$

... (4) $\quad x_1=5y_1+x_2 \qquad \dots \ (9)$

... (5) $\quad y_1=6x_2+y_2 \qquad \dots \ (10)$

... (6) $\quad x_2=7i+(17-9) \qquad \dots \ (11)$

Also, $\quad y_2=i \qquad \dots \ (12)$

Here the original equation and each of the auxiliary equations have been reduced by dividing the greater coefficient by the smaller.

The unknown quantities x and y in the original equation are given by the set of equations (7)—(12). Here we see that the quotients 3, 4, 5, 6, 7, of successive division and the assumed integer i occur in the same vertical line. It is easy to see that of the numbers r, y, x_1, y_1, each is obtained by multiplying the corresponding quotient by the number next below it and adding to the product the number just below these two numbers. This law is expressed by Āryabhaṭa in the words *adha-upariguṇam-antyayuk.* The next number x_2 is obtained by multiplying the last quotient of successive division by an assumed integer i and adding the difference of the remainders to the product. This is stated by Āryabhaṭa in the words *s'eṣaparasparabhaktaṃ matiguṇam-agrāntare kṣiptam.** This part of Āryabhaṭa's rule precedes the statement of the above law, as one must begin from below to find the values of x and y from the above set of equations.

In the present case the divisor giving the greater remainder is greater than the other. So we have begun by dividing the former by the latter. To see how we shall have to proceed in the case when the divisor corresponding to the greater remainder is smaller than the other, let us modify the above problem by substituting the divisor 222 for 3015. If x_1 and y denote the respective quotients in the new problem, it leads to equation (2) above. To solve it we must either remove equation (7) from the set of equations (7)—(12) or retain it in the form $x_1 = x_1$. In the former case x_1 (*i.e.*, the quotient corresponding to the smaller remainder) follows y (*i.e.*, the quotient corresponding to the greater remainder) in the above set of equations whereas in the case of the original problem r precedes y. In the latter case the order of the two unknown quantities is not disturbed in the last set of equations. Hence we retain equation (7) in the form $x_1 = r_1$ which may be written as $x_1 = 0.\,y + x_1$ in order to satisfy the law noted above of obtaining the unknown quantities in the original and auxiliary equations. The divisor corresponding to the greater remainder being, in this case, smaller than

* This suggests that (17—9) which is the difference between the remainders should be set down below i (the assumed integer) in the column of quotients and the assumed integer in order to find the number x_2 according to the above law. As Bhāskara first finds the least values of x and y, he puts 0 for i and consequently replaces the last quotient of Āryabhaṭa's rule by the difference between the remainders, in the column of quotients and the assumed integer which Bhāskara takes to be zero. Thus, the process of successive division is discontinued by Āryabhaṭa when the last remainder is zero and by Bhāskara when the last remainder is unity.

the other divisor, it gives the quotient 0 when divided by the latter; and this quotient is the coefficient of y in $x_1=0.y+c_1$. Hence, in this case also the divisor giving the greater remainder should, first of all, be divided by the other divisor giving the smaller remainder. Āryabhaṭa, therefore, says *Adhikāgrabhāgahāraṃ chindyāt unāgrabhagahāreṇa* (*i.e.*, the divisor giving the greater remainder should, in every case, be divided by the divisor giving the smaller remainder).

It is easy to see that, by the process just explained, the equation $Ax-By=C$ can be solved in integers for all integral values of A, B, C.

The roots of the equation $Ax+R_1=By+R_2$, obtained by Āryabhaṭa's method, will be respectively of the forms $Bmi+a$ and $Ami+\beta$ where a and β are the least integral roots of the equation.

For, $Aa+R_1=B\beta+R_2$ by supposition.

Also $ABmi=BAmi$

$\therefore \quad A(Bmi+a)+R_1=B(Ami+\beta)+R_2$

As the least value of n is $B\beta+R_2$, it is obtained by adding the greater remainder R_2 to the product of the corresponding divisor B and the remainder left on dividing by A (*i.e.*, the divisor giving the smaller remainder) β the lower of the two numbers obtained finally from the set of equations (7)—(12), which is the general value of y and is of the form $Ami+\beta$. Hence, Āryabhaṭa says "*Unāgracchedabhājite śeṣam-adhikāgracchedaguṇaṃ dvicchedāgram* *=*adhi a āgrayutam*."

It is probably in the manner shown above that Āryabhaṭa arrived at the following rule:

> "Adhikāgrabhāgahāraṃ chindyāt ūnāgrabhāgahāreṇa
> Śeṣaparasparabhaktaṃ matiguṇam-agrāntare kṣiptam
> Adha-upariguṇitam antyayugūnāgracchedabhājite śeṣam
> Adhikāgracchedaguṇaṃ dvicchedāgram-adhikāgrayutam," †

Its meaning may be stated thus:

To find a number which, when divided by two given numbers, leaves given remainders, divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller. The divisor in this operation of division should then be divided by the remainder

* This term means the number answering to the two (*dvi*) divisors (*cheda*) and the two remainders (*agra*), *i.e.*, the required number.

† This rule, as it stands, is not applicable when the number of quotients is *even*. This defect has been removed by Brahmagupta.

given by this operation. This process should be continued so long as the remainder does not vanish. The quotients of successive division should be set down, one below the other, in a vertical line in the order in which they are obtained. Set down any assumed integer under these quotients and the difference between the given remainders under it. Then multiply the last quotient of successive division by the assumed integer and add the difference between the given remainders. Continue this process of multiplying a lower number by the one just above it and adding to the product the number just under it. Finally two numbers will be obtained, the upper one being the quotient of division of the required number by the divisor corresponding to the smaller of the given remainders and the lower one the quotient given by the other divisor. The lower number should be divided by the given divisor corresponding to the smaller of the given remainders. The remainder thus obtained, being multiplied by the divisor corresponding to the greater of the given remainders and then increased by the greater remainder, gives the least number answering to the two given divisors and the two given remainders. The implication is that the least number satisfying the given conditions can also be obtained by multiplying the remainder, obtained as the result of division of the upper number by the divisor corresponding to the greater given remainder, by the divisor corresponding to the smaller given remainder and then adding the smaller remainder to the product.

The reader will now easily see that this rule of Āryabhaṭa, which also supplies an unlimited number of integral solutions of the equation $Ax-By=C$, follows, by simple and natural steps, from the first four simple rules and does not depend on Euclid's method of finding the greatest common measure or proving by that process that two numbers have no common factor. No one will deny that Āryabhaṭa and his Indian predecessors were quite familiar with the first four simple rules. Applications of these four rules can be found in abundance in early Indian works also. Therefore, Kaye's remark that "nowhere in Indian works do we meet with any process at all that could lead up to the rule" is without any foundation whatsoever.

With reference to Heath's observation that the geometrical form of the Indian solution of the equation $xy=ax+by+c$ (given by Bhāskara), it may be pointed out that Geometry, in some shape or other, had independent existence everywhere and that it was not the monopoly of the Greeks. The nature of Indian Geometry shows that Indian mathematicians never came across a Greek geometrician or a Greek geometrical work. If the facts had been otherwise, the nature of Indian

Geometry would have changed beyond recognition. For, of all branches of mathematics the Greek treatment of Geometry would have appealed most to the extremely speculative Indian mind. If, without external help, Bhāskara could make the important suggestion of using the letters of the Devanagari alphabet for unknown quantities in Algebra * and anticipate Kepler's method of integration, † would it be impossible for him to discover a geometrical solution of an algebraical equation? If Heath finds an undoubted proof of the alleged Greek origin of Āryabhaṭa's value of π in the fact that the value is expressed in terms of the Indian numerical unit *ayuta* which happens to be the same as the Greek myriad but which has been in use in India ever since the time of the Yajurveda, *i.e.*, from before the Greek civilisation came into being, it is not a matter of wonder that the mere geometrical form of the Indian solution of an equation suggests to him the possibility of its Greek origin.

Summary.

From the foregoing discussion we arrive at the following conclusions:

(1) It is not likely that Diophantus solved the Pellian equation in integers in the lost Books of the *Arithmetica.*

(2) For his solution of the so-called Pellian equation Bhāskara is indebted to Brahmagupta and Āryabhaṭa and not to any other source.

(3) Kaye's low estimate of Brahmagupta is based on a very serious misreading of facts. Brahmagupta's important contributions to the solution of the Pellian equation in integers are not due to Greek influence and inspiration.

(4) Āryabhaṭa's solution of indeterminate equations of the first degree on which Bhāskara's solution of the Pellian equation partially depends was evolved, by easy and natural steps, from the first four simple rules and not from any foreign source.

(5) The hypothesis of the ultimate Greek or foreign origin of the Indian solution of the so-called Pellian equation, therefore, cannot stand.

* *Vija gaṇita* (Sudhākar Dvivedi's edition), p. 112. See also *Bulletin, Cal. Math. Soc.*, Vol. XVIII, p. 78.

† P. C. Sengupta, *Papers on Hindu Mathematics and Astronomy* (Calcutta, 1916), pp. 19 & 20.

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2

ON THE CYCLIC PROJECTIVITY

By

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We begin in Art.1 with a simple result on the united elements of a cyclic projectivity between three elements that finds an application in the theory of plane cubic curves. We then show in Art. 4 that what * is known as the Hessian of a cubic form has a deeper significance which places it in relation with the theory of the Hessian curve of a given cubic curve. We give in Art. 6 a Lemma on the general form of order n in two variables and proceed in Art. 7 to the main burden of the paper, *viz.*, a general result on a certain type of algebraic equation, the proof being based on the ideas of cyclic projectivity put together earlier in the paper. We also explain the connection of the general result with the metrical properties † of the polar curves of a given curve, and close with a particular result of considerable interest in itself.

1. Given three points A, B, C on a line or a conic, taking A′ as the harmonic of A in regard to B, C and defining similarly B′ and C′, it is known ‡ that the three pairs AA′, BB′, and CC′ are all harmonic in regard to a single pair of points U, U′ and the relation between the two triads ABC and A′B′C′ is symmetrical. Moreover the quadratic form yielding U, U′ arises as the Hessian of the cubic form yielding ABC and equally as the Hessain of the form yielding A′B′C′. The points U, U′ are in fact the double points of the projectivity ABC $\wedge$ BCA as also of the projectivity A′B′C′ $\wedge$ B′C′A′.

* *Cf.* Goursat and Hedric, Vol. I, p. 60.

† *Cf.* Clifford, 'Collected works', p. 115-118.

‡ Baker, 'Principles', Vol. I, p. 174.

2. Given a plane cubic curve it is a known * fact that if the polar conic of a point P has a double point at P′, so has the polar conic of P′ a double point at P. Such points as P,P′ then lie on another cubic called the Hessian on which they are said to constitute a pair of conjugate points; and they are the united points of involution cut out on the line PP′ by the various polar conics of the cubic.

3. Suppose now that the line PP′ meets the original cubic in points A, B, C. Then we shall prove that PP′ are the united points of the projectivity ABC $\wedge$ BCA. For, let the polar conic of A meet PP′ in the further point A′ and so also let B′ and C′ be defined as the further points common to PP′ and the polar conics of B and C respectively. Then in virtue of the definition of a polar conic A, A′ are harmonic in regard to B, C and similiarly B,B′ and C,C′ are harmonic respectively in regard to C, A and A, B. From Art. 2 it appears that PP′ are the united points of the involution of which AA′, BB′, CC′ are pairs; whence it follows from Art. 1 that they are the united points of the projectivity ABC $\wedge$ BCA as also of the projectivity A′B′C′ $\wedge$ B′C′A′.

4. Given a cubic form yielding three points A, B, C the Hessian form thereof yielding two points P, P′ is a phenomenon in the theory of forms and is independent of the theory of curves. The result proved in Art. 3 serves to show that there is a deeper significance attaching to this nomenclature, *viz.*, that there exists a cubic curve passing through A,B,C, whose Hessian curve passes through P,P′ and has these two points for conjugate points thereon.

5. The analysis may be exhibited as follows. Given three unequal constants a_1, a_2, a_3, if the equation

$$\frac{x_0-a_1}{y-a_1}+\frac{x_0-a_2}{y-a_2}+\frac{x_0-a_3}{y-a_3}=0$$

has a double root at $y=y_0$, so has the equation

$$\frac{y_0-a_1}{x-a_1}+\frac{y_0-a_2}{x-a_2}+\frac{y_0-a_3}{x-a_3}=0$$

* Hilton, 'Higher Plane Curves', p. 241

also a double root at $x=x_0$; and the two roots x_0, y_0 are roots of the quadratic

$$(z-a_1)^2(a_2-a_3)^2+(z-a_2)^2(a_3-a_1)^2+(z-a_3)^2(a_1-a_2)^2=0.$$

Calling the two roots z_1 and z_2, it is an algebraic inference that

$$\frac{3}{z_1-z_2}=\frac{1}{a_1-z_2}+\frac{1}{a_2-z_2}+\frac{1}{a_3-z_2}$$

and z_1, z_2 may herein be interchanged.

The last statement is the metrical expression of the known result that given a cubic curve and two points P, P′ such that the polar conic of either has a double point at the other, then the polar line of either point passes through the other. As a particular case we may deduce the corresponding result regarding the sides of an inflexional triangle of a cubic curve and the pairs of vertices thereon.

6. We now proceed to give indications of a generalisation* of the preceding work. We begin with the following Lemma.

The general polynomial

$$(x-c_1)(x-c_2).........(x-c_n)$$

where the c's are all different can always be exhibited in the form

$$\lambda(x-\alpha)^n-\mu(x-\beta)^n$$

in an infinite number of ways when n is 2 and in just a unique manner when n is 3 ; on the other hand when $n\geq 4$ this is not true for general values of the c_i's and is only true under the conditions comprised in the statement that for some order of the c's

$$\alpha\beta c_1c_2\ldots c_n \wedge \alpha\beta c_2\ldots c_nc_1.$$

* An early exposition of the main ideas about the general cyclic projectivity is to be found in three classical memoirs by Lüroth in the *Math. Ann.*, Vols. 8, 9, 11.

The proof of this result lies in the fact that the c_i's are to be obtained from

$$x-\alpha=\sqrt[n]{\mu/\lambda}\ (x-\beta)\rho_i, \qquad \rho_i{}^n=1$$

that is,

$$c_i=\frac{\alpha-\sqrt[n]{\mu/\lambda}\,\beta\rho_i}{1-\sqrt[n]{\mu/\lambda}\,\rho_i},$$

where the nth root has some definite value.

Because a lineo-linear transformation does not affect cross-ratios, any relation amongst the cross-ratios of the c_i's is equally true as a relation amongst the cross-ratios of the ρ_i's and vice versa.

And finally it is a fact that if ρ^n equals unity and no two of the roots ρ_i are equal, then

$$0\ \infty\ \rho_1\rho_2\ldots\ldots\rho_n \wedge 0\ \infty\rho_2\ldots\ldots\rho_n\rho_1;$$

for since ρ_i may be taken as ρ^i the projectivity that sends $1, \rho, \rho^2$ into ρ, ρ^2, ρ^3 respectively is

$$\frac{x-1}{x-\rho}=\frac{y-\rho}{y-\rho^2}$$

and this projectivity sends $x=\rho^k$ into $y=\rho^{k+1}$ and $x=\rho^{n-1}$ into $y=\rho^n=1$; and has $0, \infty$ for united points. This completes the proof of the Lemma.

7. The following is the general result that suggests itself. The equation

$$\sum_{i=1}^{n} \frac{x_0-a_i}{y-a_i}=0$$

has a $(n-1)$-fold point at $y=y_0$ if and only if

$$x_0a_1a_2\ldots a_n \wedge x_0a_2\ldots a_n\ a_1;$$

and the root is then at $y=y_0$ where y_0 is the second united point of the projectivity indicated. Under the same conditions equally is it true that the equation

$$\sum_{i=1}^{n} \frac{y_0-a_i}{x-a_i}=0$$

also has a $(n-1)$-fold root at $x=x_0$. We assume throughout that the n quantities a_i are all different.

The proof is as follows. Given the group of n points on a line

$$(x-a_1)(x-a_2)\ldots\ldots\ldots\ldots(x-a_n)$$

the first polar group of the point $x=x_0$ is given by

$$\sum_i (x_0-a_i)/(x-a_i)=0$$

and consists of $(n-1)$ points distinct in general. Now consider a curve $f(x, y, z)$ of order n meeting the line AB in the n points mentioned. The first polar curve of A is then $f_x=0$ and is a curve of order $(n-1)$. It will have a $(n-1)$-fold point at B if f_x is independent of y. Thus

$$f_x=f_{n-1}(x, z)$$

whence

$$f=x(x, z)_{n-1}+(y, z)_n$$

$$=x^n-cy^n+zx(x, z)_{n-2}+z(y, z)_{n-1}$$

$$=x^n-cy^n+zx(x, z)_{n-2}+yz(y, z)_{n-2}+dz^n;$$

and because of the symmetry as between x and y, it follows that the first polar curve of B has a $(n-1)$-fold point at A. The intersections of f with AB are thus given by

$$x^n-cy^n=0,$$

and their parameters as points on the line are ρ^i where ρ^n equals unity. We have seen in Art. 6 that for such a group of points the relation of

cyclic projectivity holds, the united points being A, B as already shown. This completes the proof of the result.

The particular case when n equals 4 is of interest in that no imaginaries are involved, and may be put down as follows. Given that the equation

$$\frac{x_0-a}{y-a}+\frac{x_0-b}{y-b}+\frac{x_0-c}{y-c}+\frac{x_0-d}{y-d}=0$$

has a triple root at $y=y_0$, the necessary and sufficient condition is that $(a, b, c, d)=-1$ and when this condition is satisfied it is equally true that the equation

$$\frac{y_0-a}{x-a}+\frac{y_0-b}{x-b}+\frac{y_0-c}{x-c}+\frac{y_0-d}{x-d}=0,$$

also has a triple root at $x=x_0$; and that if a, b and c, d are the two pairs that separate each other harmonically, then x_0, y_0 represents the single pair simultaneously harmonic with respect to both a, b and c, d. This particular result is capable of easy independent verification.

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ON A CLASS OF TRANVERSALS CUTTING THE SIDES OF A TRIANGLE

BY

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1. Some of the following three propositions are so simple that I suspect them to be known already though not to my knowledge. They can be easily proved with the help of elementary geometry.

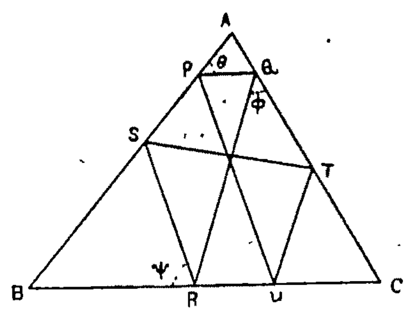
Proposition I. P is a point on the side AB of the triangle ABC. The lines PQ, QR, RS, ST, TU are drawn so that PQ is parallel to BC and cuts AC at Q, QR is parallel to AB and cuts BC at R, RS is parallel to AC and cuts AB at S, ST is parallel to BC and cuts AC at T, TU is parallel to AB and cuts BC at U, UV is parallel to AC and cuts AB at V. The point V coincides with P.

Proposition II. The proposition remains true if the lines PQ, QR, etc., instead of being parallel to the sides BC, AC, etc., be antiparallel to them.

Proposition III. The proposition will hold if PQ be parallel to the chord of contact of a circle touching AB and AC, the other lines having similar property.

The object of the present paper is to find a more general class of similar transversals and to investigate their properties.

2. Let P be a point on the side AB of the triangle ABC. The line PQ makes an angle θ with BA and cuts AC at Q, QR makes an angle ϕ with AC and cuts BC at R, RS makes an angle ψ with CB and cuts AB at S. The construction is repeated starting with S and giving rise to the points T, U, and V. To find the condition that V may coincide with P.



Let AP$=x$ so that $\dfrac{x}{\mathrm{AQ}}=\dfrac{\sin(\mathrm{A}+\theta)}{\sin\theta}$.

$$\therefore\ \mathrm{AQ}=\frac{\sin\theta}{\sin(\mathrm{A}+\theta)}x \text{ and } \mathrm{CQ}=b-\frac{\sin\theta}{\sin(\mathrm{A}+\theta)}x.$$

Again $\dfrac{\mathrm{CQ}}{\mathrm{CR}}=\dfrac{\sin(\mathrm{C}+\phi)}{\sin\phi}$

$$\because\ \mathrm{CR}=\frac{\sin\phi}{\sin(\mathrm{C}+\phi)}\left\{b-\frac{\sin\theta}{\sin(\mathrm{A}+\theta)}x\right\},$$

$$\text{and BR}=a-\frac{\sin\phi}{\sin(\mathrm{C}+\phi)}\left\{b-\frac{\sin\theta}{\sin(\mathrm{A}+\theta)}x\right\}.$$

Also $\dfrac{\mathrm{BR}}{\mathrm{BS}}=\dfrac{\sin(\mathrm{B}+\psi)}{\sin\psi}$

$$\therefore\ \mathrm{AS}=c-\frac{\sin\psi}{\sin(\mathrm{B}+\psi)}\left[a-\frac{\sin\phi}{\sin(\mathrm{C}+\phi)}\left\{b-\frac{\sin\theta}{\sin(\mathrm{A}+\theta)}x\right\}\right].$$

Putting $\dfrac{\sin\psi}{\sin(\mathrm{B}+\psi)}=p$, $\dfrac{\sin\phi}{\sin(\mathrm{C}+\phi)}=q$ and $\dfrac{\sin\theta}{\sin(\mathrm{A}+\theta)}=r$

we find BS$=c-ap+bpq-pqrx=\mathrm{L}-\mathrm{M}x$, say.

The point S is now on the same footing as P, so that if V be the second position of S we must have

$\mathrm{AV}=\mathrm{L}-\mathrm{M}(\mathrm{L}-\mathrm{M}x)=\mathrm{L}-\mathrm{LM}+\mathrm{M}^2x=x$ if V coincides with P.

$$\therefore\ \mathrm{L}(1-\mathrm{M})=x(1-\mathrm{M}^2).$$

This relation must hold for all values of x. It must therefore be an identity which is obviously the case if M$=1$, *i.e.*, $pqr=1$.

Giving to p, q and r their proper values the condition becomes

$\sin(\mathrm{A}+\theta)\sin(\mathrm{B}+\psi)\sin(\mathrm{C}+\phi)=\sin\theta\sin\phi\sin\psi$.

In Prop. I, $\theta=\mathrm{B}$, $\phi=\mathrm{A}$ and $\psi=\mathrm{C}$.

In Prop. II, $\theta=\mathrm{C}$, $\phi=\mathrm{B}$ and $\psi=\mathrm{A}$.

In Prop. III, $\theta=\frac{\pi-A}{2}$, $\phi=\frac{\pi-C}{2}$ and $\psi=\frac{\pi-B}{2}$.

Hence we enunciate the following general proposition:

If from a point P on the side AB of the triangle ABC we draw a line PQ making an angle θ with BA and cutting AC at Q if QR and RS be similarly drawn making $\angle s$ ϕ and ψ with AC and CB, then we shall come back to the starting point after a single repetition if

$$\sin(A+\theta)\sin(B+\psi)\sin(C+\phi)=\sin\theta\sin\phi\sin\psi \qquad \dots \quad (1)$$

3. Again since $AS=C-ap-bpq-pqrx$, by equating this to x, the point S, for all values of p, q and r may be made to coincide with P, so that instead of six lines forming a hexagon we get a triangle inscribed in the triangle ABC.

In Prop. I, it is the triangle formed by the mid points of the sides, in Prop. II, it is the pedal triangle. Both these triangles are homological with ABC.

We shall now prove that in the general case also the triangle thus formed will be homological with ABC.

Let $AP=t$

If S coincides with P, the trilinear co-ordinates of P, Q and R will be

$$[bp\{a-q(b-rt)\},\ at,\ 0],\ [c(b-rt),\ 0,\ art],$$
$$\text{and } [0,\ cq(b-rt),\ b\{a-q(b-rt)\}].$$

Forming now the equations of AR, BQ and CP, the condition that these lines may meet at a point is found to be

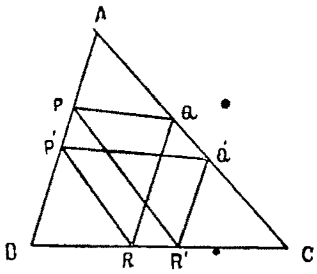
$$\begin{vmatrix} at, & -bp\{a-q(b-rt)\}, & 0 \\ art, & 0, & -c(b-rt) \\ 0 & -b\{a-q(b-rt)\}, & cq(b-rt) \end{vmatrix}=0$$

$$\text{or, } pqr=1$$

Hence we get a geometrical interpretation of (1), *viz.*,

The lines PQ, QR and RS are parallel to the sides of a triangle inscribed in ABC and homological with it.

4. Let us now denote the second series of points by P′, Q′ and R′. Putting $AP=a_1$, $AQ=a_2$, $CQ=c_1$, $CR=c_2$, $BR=b_1$ and $BP'=b_2$, the condition (1) reduces to



$$a_1b_1c_1=a_2b_2c_2 \qquad \ldots \quad (2)$$

When the hexagon reduces to a triangle, P′ coincides with P, so that $b_2=c-a_1$ and the condition reduces to

$$\left(1-\frac{b}{c_1}\right)\left(1-\frac{c}{a_1}\right)\left(1-\frac{a}{b_1}\right)+1=0.$$

5. Another geometric interpretation of (1) and (2).

Let us take the hexagon as before, the condition $a_1b_1c_1=a_2b_2c_2$ gives

$$AP.\ BR.\ CQ.=AQ.\ CR.\ BP'.$$

As the position of P′ is similar to that of P, starting from P′ we get

$$AP'.\ BR'.\ CQ'=AQ'.\ CR'.\ BP.$$

$$\therefore AP.\ AP'.\ BR.\ BR'.\ CQ.\ CQ'.=AQ.\ AQ'.\ CR\ \ CR'.\ BP.\ BP'.$$

Hence from Carnot's theorem we see that the six points P, Q, R, P′, Q′, and R′ lie on a conic.

If we calculate the barycentric co-ordinates of P, Q and R under the condition $a_1b_1c_1=a_2b_2c_2$, the equation of PQ is found to be

$$\begin{vmatrix} x & y & z \\ c-a_1 & a_1 & o \\ c_1 & o & b-c_1 \end{vmatrix}=0,$$

or $$a_1(b-c_1)x-(b-c_1)(c-a_1)y-c_1a_1z=0.$$

Similarly the equations of QR and RP′ are found to be

$$-(a-b_1)(b-c_1)x-b_1c_1y+c_1(a-b_1)z=0$$

and

$$-(c-b_2)x+b_1b_2y-b_2(a-b_1)z=0 \quad \text{respectively.}$$

$\therefore$ the points of intersection of PQ, QR and RP′ with BC, CA and AB respectively will lie on a straight line if

$$\begin{vmatrix} o & -c_1 a_1 & (b-c_1)(c-a_1) \\ c_1 & o & b-c_1 \\ b_2 & c-b_2 & o \end{vmatrix} = 0$$

i.e.,
$$\left(1-\frac{c}{a_1}\right)\left(1-\frac{c}{b_2}\right)=1.$$

When the hexagon reduces to a triangle $c-a_1=b_2$ and $c-b_2=a_1$, so that this condition is satisfied. As the two triangles are in perspective we get another proof that when the hexagon degenerates into a triangle, the triangle thus formed is always in perspective with ABC.

6. We shall now give a third and more elegant geometric interpretation.

The lines PQ, QR and RS are parallel to the chords of contact of an inscribed conic.

With the previous notation the co-ordinates of P, Q, R, and P′ are

$$\left(\frac{c-a_1}{c}, \frac{a_1}{c}, o\right), \quad \left(\frac{c_1}{b}, o, \frac{b-c_1}{b}\right), \quad \left(o, \frac{c_2}{a}, \frac{a-c_2}{a}\right)$$

$$\text{and } \left(\frac{b_2}{c}, \frac{c-b_2}{c}, o\right) \quad \text{respectively.}$$

Writing the general equation of an inscribed conic in the forms

$$(\mu y+\nu z-\lambda x)^2=4\mu\nu yz, \quad (\nu z+\lambda x-\mu y)^2=4\nu\lambda zx$$
$$\text{and } (\lambda x+\mu y-\nu z)^2=4\lambda\mu xy,$$

the three chords of contact are found to be

$$\mu y+\nu z-\lambda x=0, \quad \nu z+\lambda x-\mu y=0 \text{ and } \lambda x+\mu y-\nu z=0.$$

Since PQ passes through P and is parallel to $\mu y+\nu z-\lambda x=0$, its equation is

$$(\lambda+\mu)a_1 x+(\lambda+\mu)(c-a_1)y+\{c(\nu+\lambda)-a_1(\lambda+\mu)\}z=0.$$

Putting $y=0$, the co-ordinates of Q are found to be

$$\frac{(\nu+\lambda)c-(\lambda+\mu)a_1}{(\nu+\lambda)c}, \quad 0, \quad \frac{(\lambda+\mu)a_1}{(\nu+\lambda)c}.$$

In the same way the co-ordinates of R and P′ are found to be

$$0, \quad \frac{(\nu+\lambda)c_1}{(\mu+\nu)b}, \qquad \frac{(\mu+\nu)b-(\nu+\lambda)c_1}{(\mu+\nu)b}$$

and

$$\frac{(\mu+\nu)b_1}{(\lambda+\mu)a_1}, \quad \frac{(\lambda+\mu)a-(\mu+\nu)b_1}{(\lambda+\mu)a}, \quad 0 \text{ respectively.}$$

Comparing these with the co-ordinates found already we have

$$\frac{a_2}{b}=\frac{(\lambda+\mu)a_1}{c}, \quad \frac{b_2}{c}=\frac{(\mu+\nu)b_1}{(\lambda+\mu)a} \text{ and } \frac{c_2}{a}=\frac{(\nu+\lambda)c_1}{b}.$$

$$\because \frac{a_1}{a_2}=\frac{c(\nu+\lambda)}{b(\lambda+\mu)}, \quad \frac{b_1}{b_2}=\frac{a(\lambda+\mu)}{c(\mu+\nu)}, \quad \frac{c_1}{c_2}=\frac{b(\mu+\nu)}{a(\nu+\lambda)}.$$

$$\therefore \quad \frac{a_1b_1c_1}{a_2b_2c_2}=1.$$

$\therefore$ the lines parallel to the chords of contact of an inscribed conic form a closed set of six lines.

The converse of this can be proved without any difficulty.

7. By comparing the equations $l_r x+m_r y+n_r z=0$ $(r=1, 2, 3)$ with $\mu y+\nu z-\lambda x=0$, etc., the condition that PQ, QR, and RP may have their equations $l_1 x+m_1 y+n_1 z=0$, etc., is found to be

$$(n_1-l_1)(l_2-m_2)(m_3-n_3)+(l_1-m_1)(n_2-m_2)(n_3-l_3)=0$$

If the lines be given in the form

$$\frac{X-x_r}{l_r}=\frac{Y-y_r}{m_r}=\frac{Z-z_r}{n_r}, \quad (r=1, 2, 3)$$

the condition becomes

$$\begin{vmatrix} -l_1 & m_1 & n_1 \\ l_2 & -m_2 & n_2 \\ l_3 & m_3 & -n_3 \end{vmatrix}=0.$$

Further if the equations of PQ and QR only be given, the equation of RP′ will be

$$\begin{vmatrix} x & -y & z \\ m_1-n_1 & n_1-l_1 & l_1-m_1 \\ m_2-n_2 & n_2-l_2 & l_2-m_2 \end{vmatrix} = 0$$

when the hexagon reduces to a triangle these lines become the actual chords of contact and we get the proposition

"If a triangle be inscribed in another triangle and be homologous to it, a conic may be inscribed in the second and circumscribed to the first."

In Prop. I and II, these conics are

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0$$

and

$$\sqrt{x \cot A} + \sqrt{y \cot B} + \sqrt{z \cot C} = 0,$$

In Prop. III, it is the incircle.

8. In all these propositions the points P, Q etc., are taken on the sides of a triangle but similar theorems hold in the cases when the points are taken on certain curves. A few propositions of this nature are given below.

From a point A on a circle, a line AB is drawn parallel to a given line (1) and cutting the circle again at B; from B a line is drawn parallel to a given line (2) and cutting the circle again at C; from C a line is drawn parallel to a given line (3) and cutting the circle again at A′. Starting from A′ the construction is repeated, the point A″ must coincide with A.

Proof:—If θ be the eccentric angle of A and the lines (1), (2) and (3) make angles θ_1, θ_2 and θ_3 with the axis of x, the eccentric angle of A′ will be found to be

$$\pi+2(\theta_3-\theta_2+\theta_1)-\theta.$$

A′ being on the same footing as A, the eccentric angle of A″ will be

$$\pi+2(\theta_3-\theta_2+\theta_1)-\{\pi+2(\theta_3-\theta_2+\theta_1)-\theta\}=\theta.$$

Hence the proposition. Projecting we may deduce Pascal's Theorem.

Another proposition of the same type is the following:—

AB is a chord of a circle making an angle θ with the tangent at A, the chord BC makes an angle ϕ with the tangent at B, and cuts the circle again at C, the chord CA′ makes an angle ψ with the tangent at C and cuts the circle again at A′. The construction is repeated starting with A′. The point A″ must coincide with A.

It is easily seen that if the vectorial angle of A is Ω, the vectorial angle of A′ is $2(\theta-\phi+\psi)-\Omega$, whence it is at once inferred that A″ will coincide with A.

9. We shall now take the points on cubic curves and deduce similar propositions.

Through a point P on the cubic $\alpha\beta\gamma+k(a\alpha+b\beta+c\gamma)^3=0$, PQ is drawn parallel to an inflexional asymptote cutting the cubic at Q, QR is drawn parallel to a second asymptote cutting the cubic again at R, RP′ is drawn parallel to a third asymptote and cuts the cubic at P′. The construction is repeated starting with P′. The point P″ thus obtained will coincide with P.

Let P be the point $(\alpha_1, \beta_1, \gamma_1,)$. It is easily seen that the sides of the triangle of reference are the asymptotes.

The equation of a line through P parallel to BC is

$$\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta-c\alpha_1\gamma=0.$$

$$\therefore \gamma=\frac{\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta}{c\alpha_1}$$

Substituting in the equation of the cubic, for the point Q we have

$$\alpha\beta\frac{\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta}{c\alpha_1}+k\left\{a\alpha+b\beta+\frac{\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta}{\alpha_1}\right\}^3=0.$$

$$\therefore \alpha\beta\alpha_1^2\{\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta\}+ck\{\alpha(a\alpha_1+b\beta_1+c\gamma_1)\}^3=0.$$

$$\therefore \alpha\beta\alpha_1^2\{\alpha(b\beta_1+c\gamma_1)-b\alpha_1\beta\}-c\alpha^3.\ \alpha_1\beta_1\gamma_1=0 \quad \because \text{P lies on the cubic.}$$

$$\therefore \alpha^2c\beta_1\gamma_1-\alpha\beta\alpha_1\ (b\beta_1+c\gamma_1)+\beta^2b\alpha_1^2=0.$$

$\therefore$ for the points P and Q

$$\Pi\frac{\alpha}{\beta}=\frac{b\alpha_1^2}{c\beta_1\gamma_1}$$

But for the point P, $\frac{\alpha}{\beta}=\frac{\alpha_1}{\beta_1}$.

$\therefore$ for the point Q, $\frac{\alpha}{\beta}=\frac{b\alpha_1}{c\gamma_1}$.

In the same way for the point Q, $\frac{\alpha}{\gamma}=\frac{c\alpha_1}{b\beta_1}$

$\therefore$ for the point Q, $\alpha:\beta:\gamma=bc\alpha_1:c^2\gamma_1:b^2\beta_1=\alpha_2:\beta_2:\gamma_2$, say.

In the same way for R, $\alpha:\beta:\gamma=b^2\beta_2:a^2\alpha_2:ab\gamma_2=\alpha_3:\beta_3:\gamma_3$

For P′, $\alpha:\beta:\gamma=c^2\gamma_3:ca\beta_3:a^2\alpha_3=\alpha_4:\beta_4:\gamma_4$.

Expressing α_3, β_3 and γ_3 in terms of α_1, β_1 and γ_1 we have

$$\alpha_4:\beta_4:\gamma_4=b^2\beta_1:a^2\alpha_1:ab\gamma_1.$$

For P″, $\alpha_7:\beta_7:\gamma_7=b^2\beta_4:a^2\alpha_4:ab\gamma_4=a^2b^2\alpha_1:a^2b^2\beta_1:a^2b^2\gamma_1$

$$=\alpha_1:\beta_1:\gamma_1.$$

$\therefore$ P″ must coincide with P.

If we take the cubic $\alpha\beta\gamma+(l\alpha+m\beta+n\gamma)(a\alpha+b\beta+c\gamma)^2=0$ and perform the above construction the same result follows.

This may be deduced analytically as before or we may prove it by the Theory of Residuation.

10. A point P on a cubic is joined to a real point of inflexion and cuts the cubic at Q which joined to a second real point of inflexion cuts the cubic at R which again joined to a third real point of inflexion cuts the cubic at P′. The construction is repeated starting with P′. The point P″ must coincide with P.

If we take the canonical form of the cubic, *viz.*,

$$x^3+y^3+z^3+6mxyz=0,$$

the three real points of inflexion L, M and N are

$$(0, 1, -1), (-1, 0, 1) \text{ and } (1, -1, 0).$$

Let P be the point (x_1, y_1, z_1).

The equation of AL is

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ 0 & 1 & -1 \end{vmatrix} = 0 \text{ or } x = -\frac{x_1(y+z)}{y_1+z_1} \qquad \ldots \quad (1)$$

$$\therefore -\frac{x_1^3(y+z)^3}{(y_1+z_1)^3} + y^3 + z^3 - 6myz\,\frac{x_1(y+z)}{y_1+z_1} = 0$$

In this equation the co-efficient of y^3 is the same as the co-efficient of z^3.

$$\therefore \Pi\ \frac{y}{z} = -1.$$

But one value of $\frac{y}{z}$ is $\frac{y_1}{z_1}$ and the other value is -1.

$\therefore$ for the point Q, $\frac{y}{z} = \frac{z_1}{y_1}$

whence from (1) we get for Q, $\frac{x}{x_1} = \frac{y}{z_1} = \frac{z}{y_1}$ *i.e.*, $\frac{x}{x_2} = \frac{y}{y_2} = \frac{z}{z_2}$, say.

In the same way for R, $\frac{x}{z_2} = \frac{y}{y_2} = \frac{z}{x_2}$ *i.e.*, $\frac{x}{x_3} = \frac{y}{y_3} = \frac{z}{z_3}$, say.

Similarly for P′, $\frac{x}{y_3} = \frac{y}{x_3} = \frac{z}{z_3}$.

Expressing x_3, y_3 and z_3 in terms of x_1, y_1 and z_1 we have for P′.

$$\frac{x}{z_1} = \frac{y}{y_1} = \frac{z}{x_1}$$

Thus when we traverse the cubic only once the first and the third co-ordinates are interchanged while the second co-ordinate remains unchanged.

$\therefore$ after traversing twice we get the same set of co-ordinates,

i.e., P″ coincides with P.

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On the complex multiplication of elliptic functions with imaginary moduli.

By

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Introduction.

1. The object of the present paper is to give several cases of complex multiplication of elliptic functions with imaginary moduli, by means of a method first used by Cayley * for the transformation of the third order.

The method of Cayley does not appear to have been used by any subsequent writer. For this reason the results given by me are believed to be of some interest.

2. Consider the differential expression $\frac{Mdy}{\sqrt{(1-y^2)(1-v^8y^2)}}$.

Write $y=\frac{U}{V}$ where U, V are rational and integral functions of x, one of them of order p and the other of order p or $p-1$. It is well-known that the co-efficients in U, V may be so determined as to lead to an equation

$$\frac{Mdy}{\sqrt{(1-y^2)(1-v^8y^2)}}=\frac{dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

* *Proc. L.M.S.*, series I, Vol. XIX.

Modulus Corresponding to $\triangle=5$.

3. In the Jacobian notation, the modular equation of the 5th order is

$$u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0.$$

Complex multiplication will take place if $u^8=v^8$, or $v=\gamma u$, where γ is an eighth root of unity.

Putting $v=\gamma u$ in the modular equation, we get

$$4\gamma^5u^8+(\gamma^6+5\gamma^4-5\gamma^2-1)u^4-4\gamma=0.$$

(*a*) Let,

$$\gamma^4=1.$$

Then the equation becomes

$$\gamma u^8+(1-\gamma^2)u^4-\gamma=0.$$

If $\gamma^2=1$, $u^8=1$ and we have a non-elliptic case.

Similarly for $\gamma^2=-1$, $u^8=-1$ and we have the case of the lemniscate function.

(*b*) If $\gamma^4=-1$, we can take $\gamma^2=i$.

Then we get the equation

$$4\gamma u^8+6(1+\gamma^2)\,u^4+4\gamma=0.$$

Therefore

$$u^4=-\frac{(1+i)}{2\gamma},\ -\ \frac{(1+i)}{\gamma}.$$

But the quintic transformation is given by

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\ \left(\frac{1-\alpha x+\beta x^2}{1+\alpha x+\beta x^2}\right)^2$$

where

$$2a+1=\frac{1}{M}=\frac{v-u^5}{v(1-uv^5)};\quad \beta^2=\frac{u^{10}}{v^2}.$$

(*i*) When

$$u^4=-\frac{(1+i)}{2\gamma}$$

we get

$$\beta^2=\frac{1}{2i}\ ,\quad 2a+1=\frac{1}{M}=-(2i-1).$$

Therefore

$$\frac{1+\operatorname{sn}(2i-1)u}{1-\operatorname{sn}(2i-1)u}=\frac{1-\operatorname{sn}\,u}{1+\operatorname{sn}\,u}\left\{\frac{1+i\operatorname{sn}\,u-\frac{(1-i)}{2}\operatorname{sn}^2u}{1-i\operatorname{sn}\,u-\frac{(1-i)}{2}\operatorname{sn}^2u}\right\}^2$$

(*ii*) When

$$u^4=-\frac{(1+i)}{\gamma}$$

we have

$$\beta^2=-2i\,,\ 2a+1=\frac{1}{M}=-(2i+1).$$

Therefore

$$\frac{1+\operatorname{sn}\,(2i+1)u}{1-\operatorname{sn}\,(2i+1)u}=\frac{1-\operatorname{sn}u}{1+\operatorname{sn}u}\left(\frac{1+(1+i)\operatorname{sn}u-(1-i)\operatorname{sn}^2u}{1-(1+i)\operatorname{sn}u-(1-i)\operatorname{sn}^2u}\right)^2.$$

Modulus corresponding to $\Delta=7$.

4. The Modular equation of the 7th order is

$$(1-u^8)\,(1-v^8)=(1-uv)^8\,.$$

Putting $v=\gamma u$, we get the equation

$$8(\gamma u^2+\gamma^7u^{14})-28(\gamma^2u^4+\gamma^6u^{12})+56(\gamma^3u^6+\gamma^5u^{10})$$
$$-70\gamma^4u^8-2u^8=0.$$

(*a*) If we take $\gamma^4=1$, we have

$$2(\gamma u^2+\gamma^7u^{14})-7(\gamma^2u^4+\gamma^6u^{12})+14(\gamma^3u^6+\gamma^5u^{10})-18\gamma^4u^8=0.$$

Putting $\gamma u^2=z$, we have

$$2\left(z^3+\frac{1}{z^3}\right)-7\left(z^2+\frac{1}{z^2}\right)+14\left(z+\frac{1}{z}\right)-18=0.$$

Putting $z+\dfrac{1}{z}=y$, we get

$$2y^3-7y^2+8y-4=0,$$

or
$$(y-2)(2y^2-3y+2)=0.$$

The quadratic factor gives

$$y=\frac{3\pm\sqrt{7}i}{4}.$$

Taking the upper sign we get

$$\gamma u^2=z=\frac{1+\sqrt{7}i}{2}\text{ and }\frac{1-\sqrt{7}i}{4}.$$

The septic transformation is given by

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-ax+\beta x^2-\gamma_1x^3}{1+ax+\beta x^2+\gamma_1x^3}\right)^2$$

where

$$2a+1=\frac{1}{M},\quad 2\beta=u^3v^3\left(\frac{1}{M}-\frac{u^4}{v^4}\right),\quad \gamma_1=\frac{u^7}{v},$$

and $$\frac{1}{M}=-\frac{7u(1-uv)\,(1-uv+u^2v^2)}{u-v^7}.$$

(*i*) When $\gamma u^2=\dfrac{1+\sqrt{7}i}{2}$ we have

$$\alpha=-\frac{(1+\sqrt{7}i)}{2},\qquad 2\beta=(3\sqrt{7}i-1),\quad \gamma_1=-\frac{(5+\sqrt{7}i)}{2},$$

and $$\frac{1}{M}=-\sqrt{7}i.$$

Therefore

$$\frac{1+\operatorname{sn}\sqrt{7}iu}{1-\operatorname{sn}\sqrt{7}iu}=\frac{1-\operatorname{sn}u}{1+\operatorname{sn}u}$$

$$\times\left\{\frac{1+\dfrac{(1+\sqrt{7}i)}{2}\operatorname{sn}u+\dfrac{(3\sqrt{7}i-1)}{2}\operatorname{sn}^2u+\dfrac{(5+\sqrt{7}i)}{2}\operatorname{sn}^3u}{1-\dfrac{(1+\sqrt{7}i)}{2}\operatorname{sn}u+\dfrac{(3\sqrt{7}i-1)}{2}\operatorname{sn}^2u-\dfrac{(5+\sqrt{7}i)}{2}\operatorname{sn}^3u}\right\}^2.$$

(*ii*) When $\gamma u^2=\dfrac{1-\sqrt{7}i}{4}$, we have

$$\alpha=-\frac{(1+\sqrt{7}i)}{2},\quad \beta=\frac{3+\sqrt{7}i}{8},\ \gamma_1=\frac{\sqrt{7}i-5}{16}.$$

Therefore

$$\frac{1+\operatorname{sn}\sqrt{7}iu}{1-\operatorname{sn}\sqrt{7}iu}=\frac{1-\operatorname{sn}u}{1+\operatorname{sn}u}$$

$$\times\left\{\frac{1+\dfrac{(1+\sqrt{7}i)}{2}\operatorname{sn}u+\dfrac{(3+\sqrt{7}i)}{8}\operatorname{sn}^2u+\dfrac{(5-\sqrt{7}i)}{16}\operatorname{sn}^3u}{1-\dfrac{(1+\sqrt{7}i)}{2}\operatorname{sn}u+\dfrac{(3+\sqrt{7}i)}{8}\operatorname{sn}^2u-\dfrac{(5-\sqrt{7}i)}{16}\operatorname{sn}^3u}\right\}^2.$$

(*b*) If $\gamma^4=-1$, the equation becomes

$$2(1+\gamma^6u^{12})-7(\gamma u^2+\gamma^5u^{10})+14(\gamma^2u^4+\gamma^4u^8)$$
$$-17\gamma^3u^6=0.$$

Putting $\gamma u^2=z$ and $z+\dfrac{1}{z}=y$, we get the equation

$$(y-1)^2(2y-3)=0.$$

The first factor corresponds to $\Delta=3$, while the second factor corresponds to $\Delta=7$.

Since $y=\dfrac{3}{2}$, we have

$$z^2-\frac{3z}{2}+1=0,$$

or
$$\gamma u^2=z=\frac{3\pm\sqrt{7}i}{4}.$$

If we take $\gamma u^2=\dfrac{3+\sqrt{7}i}{4}$, we get

$$u^8=-\left(\frac{3+\sqrt{7}i}{4}\right)^4,\quad \frac{1}{M}=\sqrt{7}i,\quad \alpha=\frac{(\sqrt{7}i-1)}{2},$$

$$\beta=-\frac{(11+\sqrt{7}i)}{8},\quad \gamma_1=\frac{9-5\sqrt{7}i}{16}.$$

Therefore

$$\frac{1-\mathrm{sn}\sqrt{7}iu}{1+\mathrm{sn}\sqrt{7}iu}=\frac{1-\mathrm{sn}\ u}{1+\mathrm{sn}\ u}$$

$$\times\left\{\frac{1-\dfrac{(\sqrt{7}i-1)}{2}\mathrm{sn}\,u-\dfrac{(11+\sqrt{7}i)}{8}\mathrm{sn}^2u-\dfrac{(9-5\sqrt{7}i)}{16}\mathrm{sn}^3u}{1+\dfrac{(\sqrt{7}i-1)}{2}\mathrm{sn}\,u-\dfrac{(11+\sqrt{7}i)}{8}\mathrm{sn}^2u+\dfrac{(9-5\sqrt{7}i)}{16}\mathrm{sn}^3u}\right\}^2$$

Some other moduli.

5. In each of the following cases, I shall simply indicate the method and give the result. The process of finding the result is the same as in the preceding Arts.

(a) *Modulus corresponding to* $\Delta=11$.

This is obtained from the modular equation of the 11th order.

Putting $\gamma u^3=z$, $\gamma^2=i$, and $z-\dfrac{1}{z}=iy$, we get the equation

$$(2y+3)^2(y^3-3y^2-y+7)=0.$$

The second factor corresponds to $\Delta=11$.

Making use of the expression*

$$nM^2=\frac{\lambda\lambda'^2}{kk'^2}\,\frac{dk}{d\lambda}=\frac{v(1-v^8)}{u(1-u^8)}\,\frac{du}{dv}$$

we find that $M^2=-\dfrac{1}{11}$.

(b) *Moduli corresponding to* $\Delta=15$ *and* $\Delta=19$.

These moduli are obtained from the modular equations of the 19th order.

Let $\gamma u^4=z$, $z-\dfrac{1}{z}=y$ and $\gamma^4=1$.

We get the equation

$$\left(4y^4+\frac{17+21\sqrt{5}}{2}y^2+64\right)\left(4y^4+\frac{17-21\sqrt{5}}{2}y^2+64\right)=0.$$

* *Jacobi-Fundamenta Nova*, p. 75.

Therefore

$$8y^2 = -\frac{(17+21\sqrt{5})}{2} \pm \frac{i}{2}(17\sqrt{3}-7\sqrt{15})$$

and
$$8y^2 = -\frac{(17-21\sqrt{5})}{2} \pm \frac{i}{2}(17\sqrt{3}+7\sqrt{15}).$$

Both these values correspond to $\Delta = 15$.

Again putting, $\gamma u^2 = z$, $z - \frac{1}{z} = ix$ and $\gamma^2 = i$.

We get the equation

$$(x+1)^2(x^3 - x^2 + 3x + 9)(4x^2 - 2x - 11)^2 = 0.$$

The cubic factor corresponds to $\Delta = 19$ and $M^2 = -\frac{1}{19}$. The equation $4x^2 - 2x - 11 = 0$ corresponds to $\Delta = 15$, and gives

$$x = \frac{1 \pm 3\sqrt{5}}{4}.$$

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On the summability (C, 1) and strong summability (C, 1) of certain divergent Legendre series

BY

H. P. BANERJEA

It is well known * that the Fourier series corresponding to a continuous function is convergent (C, 1). A *direct verification* of this theorem for the case of a continuous function whose Fourier series is divergent at a point was first given † by Fejér. The corresponding theorem for the Legendre series was first proved by Haar ‡ and subsequently discussed by Chapman, § Gronwall, || Lukàcs ¶ and Fejér.** The *direct verification* of the convergence (C, 1) of certain continuous functions whose Legendre series diverges has been taken up in this paper. I prove in Art. 1, that the Legendre series corresponding to the first function given by Lukács †† is convergent (C, 1). In Art. 2, I prove that the second function ‡‡ of Lukàcs also possesses this property. In Art. 3, I show that the series corresponding to both the functions of

* "Untersuchungen über Fouriersche Reihen"—L. Fejér (*Math. Ann.*, Bd. 58, 1904). See also *Comptes Rendus* t. 131 (1900) and t. 134 (1904).

† "Sur les singularités de la série de Fourier des fonctions continues"—L. Fejér (*L'Ecole normale superieure*, tome 88, 1911).

‡ "Über die Legendresche Reihe"—A. Haar (*Rend. del Circ. Mat. di Palermo*, tomo 32, 1911). See also *Math. Ann.*, Vol. 69, 1910.

§ *Quarterly Journal of Pure and Applied Mathematics*, Vol. 43, 1912; *Math. Ann.* Bd. 72.

|| *Mathematische Annalen*, Bd. 74, 1913.

¶ *Comptes Rendus*, t. 157, 1913; *Math. Zeitschrift* Bd. 14.

** *Mathematische Zeitschrift*, Bd. 24, 1926.

†† *Mathematische Zeitschrift*, Bd. 14, 1922.

‡‡ *Ibid.*

Lukàcs are strongly summable (C, 1).* In Art. 4, I discuss for increasing n the infinitary behaviour of

$$\sum_{m=1}^{n} | s_m - s |^q, \quad \text{for } q > 1,$$

where s_m is the mth partial sum of the Legendre series and s the value of the function at the point considered.

My best thanks are due to Professor Ganesh Prasad for encouragement and interest.

1. *Lukàcs's first function.*

Let
$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{f_{\nu_n}(\theta)}{n^2}, \qquad \dots \quad (1)$$

where $f_\nu(\theta) = \sqrt{\nu+1}\ \{P_\nu(\cos\theta) - P_{\nu+2}(\cos\theta)\}\ (1-\cos\theta)$,

and $\nu_n = n^6$.

It has been proved † by Fejér that

$$| \sqrt{\nu+1}\ \{P_\nu(\cos\theta) - P_{\nu+2}(\cos\theta)\} | < C, \qquad \dots \quad (2)$$

for all integral values of ν, C being a constant independent of ν and θ.

Hence

$$\Phi(\theta) \equiv \sum_{n=1}^{\infty} \frac{\sqrt{n^6+1}}{n^2}\ \{P_{n^6}(\cos\theta) - P_{n^6+2}(\cos\theta)\}(1-\cos\theta),$$

is, on account of the inequality (2), easily seen to be a continuous function of θ, being the sum of an absolutely, and therefore uniformly, convergent series.

* In *Comptes Rendus*, t. 156, 1913, Hardy and Littlewood have proved that under certain restrictions if the Fourier series corresponding to $f(x)$ is convergent (C, 1), it is also strongly summable (C, 1). In *Fundamenta Mathematicae*, Vol. 10, 1927, A. Zygmund proves the corresponding theorem for normalised orthogonal functions. Thus Art. 3 is a direct verification of the theorem, corresponding to Hardy and Littlewood's theorem, for the Legendre series.

† *Mathematische Zeitschrift*, Bd. 24, 1926.

The kth partial sum s_k of the Legendre series of $\Phi(\theta)$ is

$$s_k(\theta)=\sum_{r=0}^{k}\frac{2r+1}{2}\,P_r(\cos\theta)\int_0^{\pi}P_r(\cos\phi)\,\Phi(\phi)\sin\phi\,d\phi$$

$$=\frac{k+1}{2}\int_{-1}^{1}\Phi(\phi)\,dx\left\{\frac{P_k(x)P_{k+1}(\cos\theta)-P_{k+1}(x)P_k(\cos\theta)}{\cos\theta-x}\right\}$$

(putting $x=\cos\phi$)

$$=\frac{k+1}{2}\sum_{n=1}^{\infty}\frac{\sqrt{n^6+1}}{n^2}\int_{-1}^{1}\frac{P_k(x)P_{k+1}(\cos\theta)-P_{k+1}(x)P_k(\cos\theta)}{\cos\theta-x}$$

$$\times\{P_{n^6}(x)-P_{n^6+2}(x)\}(1-x)\,dx$$

Hence putting $\theta=0$,

$$s_k(0)=\frac{k+1}{2}\sum_{n=1}^{\infty}\frac{\sqrt{n^6+1}}{n^2}\int_{-1}^{1}\{P_k(x)-P_{k+1}(x)\}\{P_{n^6}(x)-P_{n^6+2}(x)\}\,dx$$

So we have

$$s_{n^6-1}=-n^6\cdot\frac{\sqrt{n^6+1}}{n^2}\cdot\frac{1}{2n^6+1},$$

$$s_{n^6}=(n^6+1)\,\frac{\sqrt{n^6+1}}{n^2}\cdot\frac{1}{2n^6+1},$$

$$s_{n^6+1}=(n^6+2)\,\frac{\sqrt{n^6+1}}{n^2}\cdot\frac{1}{2n^6+5},$$

$$s_{n^6+2}=-(n^6+3)\,\frac{\sqrt{n^6+1}}{n^2}\cdot\frac{1}{2n^6+5},$$

for all integral values of n. For all other positive values of k, such that k is not of the forms $n^6\pm1$, n^6 and n^6+2, ($n=1, 2, 3, \ldots$) the values of $s_k(0)$ are all zero.

Hence, although the values of $s_k(0)$ are either in absolute value very large or zero as k becomes bigger and bigger,

$$\left| \frac{\sum_{n=0}^{k} s_n}{k+1} \right| \leq \sum_{n=1}^{N} \left\{ \frac{-n^6}{n^2} \cdot \frac{\sqrt{n^6+1}}{2n^6+1} + \frac{(n^6+1)}{n^2} \frac{\sqrt{n^6+1}}{2n^6+1} \right.$$

$$\left. + \frac{n^6+2}{n^2} \cdot \frac{\sqrt{n^6+1}}{2n^6+5} - \frac{n^6+3}{n^2} \cdot \frac{\sqrt{n^6+1}}{2n^6+5} \right\} \Big/ (k+1),$$

N being so chosen that $N^6 \leq k+1 < (N+1)^6$.

Thus

$$\left| \frac{\sum_{n=0}^{k} s_n}{k+1} \right| \leq \frac{\sum_{n=1}^{N} \frac{\sqrt{n^6+1}}{n^2} \left\{ \frac{1}{2n^6+1} - \frac{1}{2n^6+5} \right\}}{k+1}$$

$$\leq \frac{\sum_{n=1}^{N} \sqrt{1+\frac{1}{n^6}} \; \frac{1}{n^{11}\left(1+\frac{1}{2n^6}\right)\left(1+\frac{5}{2n^6}\right)}}{k+1}.$$

Here the numerator is for large values of N of the same order as $\sum \frac{1}{n^{11}}$, and as this series is absolutely convergent, the numerator tends to a finite limit, say C′, where C′ is an absolute constant, so that

$$\lim_{k \to \infty} \left| \frac{\sum_{n=0}^{k} s_n(0)}{k+1} \right| = \lim_{k \to \infty} \frac{C'}{k+1} = 0.$$

Thus the divergent Legendre series corresponding to the continuous function $\Phi(\theta)$ given by (1) is convergent (C,1) at $\theta=0$.

2. *Lukàcs's second example.*

Let $$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{\sqrt{n!}}{n^2} \{P_{n!}(\cos\theta) - P_{n!+2}(\cos\theta)\} \quad \ldots \quad (3)$$

Here we notice that on account of the inequality (2) of the previous article, the function $\Phi(\theta)$ is a continuous function of θ and its expansion in a Legendre series gives

$$\Phi(\theta)=0+0+\frac{\sqrt{2!}}{2^2}\,P_2(\cos\theta)+0-\frac{\sqrt{2!}}{2^2}\,P_4(\cos\theta)$$

$$+0+\frac{\sqrt{3!}}{3^2}\,P_6(\cos\theta)+0-\frac{\sqrt{3!}}{3^2}\,P_8(\cos\theta)+\dots$$

The partial sum s_k of the first $(k+1)$ terms of this series for $\theta=0$ is either 0 or very large as k becomes bigger and bigger ; nevertheless

$$\frac{\sum\limits_{n=0}^{k} s_n(0)}{k+1}\leq\frac{2\sum\limits_{n=2}^{N}\frac{\sqrt{n!}}{n^2}}{k+1},$$

where N is so chosen that $N!\leqq k<(N+1)!$

Thus

$$\frac{\sum\limits_{n=0}^{k} s_n(0)}{k+1}\leq\frac{2\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{k+1}$$

$$\leq\frac{\frac{2}{k+1}}{\sqrt{N!}}\;\frac{\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{\sqrt{N!}}.$$

But we know that if $M_1, M_2,\dots M_n$ be a monotone increasing sequence of positive numbers and if $\sum a_n$ is convergent,

$$\lim_{n\to\infty}\frac{M_1a_1+M_2a_2+\dots+M_na_n}{M_n}=0.$$

Therefore

$$\lim_{k\to\infty}\frac{\sum\limits_{n=0}^{k} s_n(0)}{k+1}=\lim_{k\to\infty}\frac{\frac{2}{k+1}}{\sqrt{N!}}\;\frac{\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{\sqrt{N!}}$$

$$=0, \text{ since } k+1\geq N!+1.$$

Thus in this case, as well, the divergent Legendre series corresponding to the continuous function $\Phi(\theta)$ given by (3) is convergent (C, 1) at $\theta=0$.

3. To consider the strong summability (C, 1) of the Legendre series corresponding to Lukàcs's first function, we have, since $\Phi(0)=0$,

$$\left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| \leq \sum_{n=1}^{N} \left\{ \frac{n^6\sqrt{n^6+1}}{n^9(2n^6+1)} + \frac{(n^6+1)\sqrt{n^6+1}}{n^9(2n^6+1)} + \frac{(n^6+2)\sqrt{n^6+1}}{n^9(2n^6+5)} + \frac{\sqrt{n^6+1}\,(n^6+3)}{n^9(2n^6+5)} \right\} \Big/ k+1,$$

where as before $N^6 \leq k+1 < (N+1)^6$.

Therefore

$$\left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| \leq \frac{2\sum_{n=1}^{N} \frac{\sqrt{n^6+1}}{n^2}}{k+1}$$

$$\leq \frac{2\sum_{n=1}^{N} n\sqrt{1+\frac{1}{n^6}}}{k+1}.$$

The numerator here is for large values of N of the same order as $2\sum_{n=1}^{N} n$ *i.e.*, as $N(N+1)$.

Hence

$$\lim_{k\to\infty} \left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| = \lim_{k\to\infty} \frac{N(N+1)}{k+1}$$

$$= \lim_{N\to\infty} \frac{N(N+1)}{(N+1)^6} = 0.$$

The series is therefore strongly summable (C, 1).

To discuss the strong summability (C, 1) of the series corresponding ... the second function of Lukàcs, we notice that $\Phi(0)$ is zero and that ...al sum is positive or zero. Then since the corresponding ...onvergent (C, 1), it is also strongly summable (C, 1).

The same remarks also apply to the case of Gronwall's example (*Math. Ann.*, Vol. 74).

4. To determine the greatest integral value of q such that

$$\frac{\sum_{n=0}^{k} | s_n - \Phi(0) |^q}{k+1} = 0$$

for the first function of Lukàcs, we have

$$\frac{\sum_{n=1}^{N-1} K_n}{k+1} < \left| \frac{\sum_{n=0}^{k} | s_n |^q}{k+1} \right| \leq \frac{\sum_{n=1}^{N} K_n}{k+1},$$

where K_n denotes

$$\frac{(n^6+1)^{\frac{q}{2}}}{n^{3q}} \left\{ \frac{n^{6q}+(n^6+1)^q}{(2n^6+1)^q} + \frac{(n^6+2)^q+(n^6+3)^q}{(2n^6+5)^q} \right\} \Big/ k+1,$$

and $N^6 \leq k+1 < (N+1)^6$.

Here we see that for sufficiently large values of N, $\sum_{n=1}^{N} K_n$ is of the same order as $\sum_{1}^{N} n^q$ and this is $O(N^{q+1})$.

Therefore the required greatest value of q is 4, since we have

$$\frac{\sum_{n=0}^{k} | s_n - \Phi(0) |^4}{k+1} = O(1).$$

For the second function of Lukàcs, we have

$$\left| \frac{\sum\limits_{n=0} | s_n - \Phi(0) |^q}{k+1} \right| \propto \frac{\sum\limits_{n=2}^{N} \frac{(n!)^{\frac{q}{2}}}{n^{2q}}}{k+1},$$

$$\text{where} \quad N! \leq k < (N+1)!$$

$$\propto \frac{\left\{ \frac{(2!)^{\frac{q}{2}}}{2^{2q}} + \frac{(3!)^{\frac{q}{2}}}{3^{2q}} + \cdots \frac{(N!)^{\frac{q}{2}}}{N^{2q}} \right\}}{N!+1}$$

In the limit the right hand side becomes zero, when $\frac{q}{2}=1$. *i.e.* $q=2$.

For $q > 2$, the right hand side is infinite * in the limit. Thus the required greatest value of q is 2.

Bull. Cal. Math. Soc., Vol. XIX, No. 4 (1928).

* In virtue of Stirling's formula,

$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N (1+\epsilon), \quad \text{where} \lim_{N=\infty} \epsilon = 0.$$